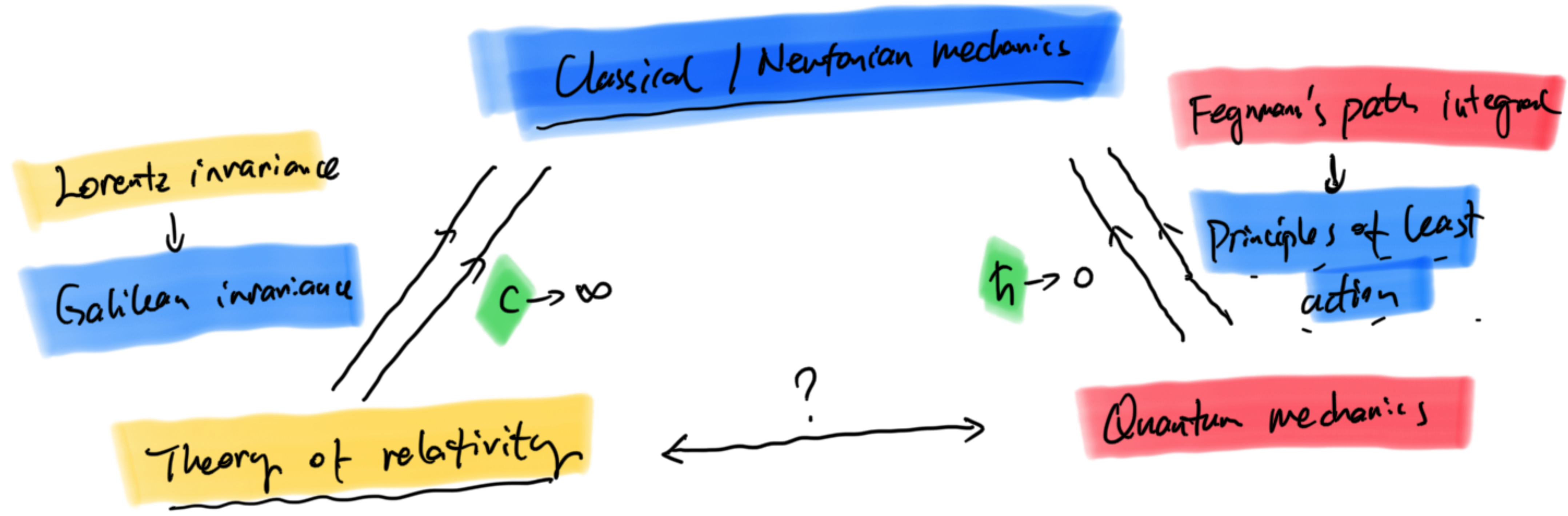
Two dimensional Strongly correlated topological system

- Quantum Hall effect (QHE)
  - Integer quantum Hall effect → Nobel prize 1985
  - Fractional quantum Hall effect → Nobel prize 1998
  - Topological physics ⋯ 2016

- Lecture 1: Quantum mechanics

  - Overview.



(A geometric theory unites time and space on equal footing)

(A linear theory where time is just a parameter)

Two fundamental constants

- The smallness of $\hbar = 1.05 \times 10^{-34}\ m^2 kg\, s^{-1}$ (angular momentum)

  energy quantum $= \hbar\omega$ ↳ frequency (s^{-1})

- The largeness of $c = 3 \times 10^{8}\ m s^{-1}$

- The quantum Hall effect — chiral Luttinger liquid — 1+1 system

- A two-dimensional system where "c" and "$\hbar$" are tunable

magnetic length

- A topological system from the quantum effect of electrons

- Interplay of geometry and topology at low temperature

  - Bulk-edge correspondence (holographic principle)

  - Lorentz invariance $\leftrightarrow$ conformal invariance

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- Hilbert space and operators

  $\swarrow$ Hermitian $\searrow$ unitary

- Let $\hat{A}$ be a Hermitian operator (representing a physical measurement)

$A = A^{+}$

  - All eigenvalues are real

$$\hat{A}|a\rangle = a|a\rangle$$

the measured value

The corresponding physical state

  - For any state

$$|\psi\rangle = \sum_a C_a |a\rangle \quad \left(\int da\, C_a |a\rangle \right)$$

amplitude

$$\langle\psi|\psi\rangle = \sum_a C_a^* C_a = 1$$

$\rightarrow$ total probability

- Let $\hat{U}$ be a unitary operator: (translation of a state)

$$\hat{U}_{\Delta a} |a\rangle = |a+\Delta a\rangle$$

$$\langle a+\Delta a | a+\Delta a\rangle = 1 = \langle a | \hat{U}^{+}_{\Delta a} \hat{U}_{\Delta a} | a\rangle$$

$$\Rightarrow \quad \hat{U}^{+}_{\Delta a} \hat{U}_{\Delta a} = I$$

Assuming "a" is continuous

$$\delta a = \lim \Delta a \to 0$$

$$|a+\delta a\rangle = |a\rangle + \delta a \cdot \partial_a |a\rangle + O(\delta a^2)$$

$$= \hat{U}_{\delta a} |a\rangle$$

$$\Rightarrow \quad \hat{U}_{\delta a} = 1 + \delta a \cdot \partial_a = e^{-i\delta a \cdot \hat{B}}$$

$$\hat{B} = -i \cdot \partial_a \rightarrow \text{Hermitian operator}$$

$$\hat{B} = \hat{B}^{+} = (-i\,\partial_a)^{+} = -i\,\partial_a$$

$$(\partial_a)^{+} = -\partial_a$$

① $$\lim_{\Delta a \to 0} \hat{U}_{\Delta a} = I$$

② $$\hat{U}^{+}_{\delta a} \hat{U}_{\delta a} = I$$

③ $$\hat{U}_{\delta a_1} \hat{U}_{\delta a_2} = \hat{U}_{\delta a_1 + \delta a_2}$$

④ $$\hat{U} = \hat{U}^{-1}$$

(4) $U_{-\delta a_1} = U_{\delta a_1}$

Lie algebra for infinitesmall transformation

- $\boxed{[\hat{A}, \hat{B}] = i}$

$\underbrace{\quad}_{\text{canonical conjugate}}$

annihilation ← $\hat{X} = \frac{1}{\sqrt{2}} (\ell \hat{A} + i \ell^{-1} \hat{B})$

creation ← $\hat{X}^{+} = \frac{1}{\sqrt{2}} (\ell \hat{A} - i \ell^{-1} \hat{B})$

ℓ ↪ a tunable parameter

} Ladder operators

↓ important building blocks for both single particle and interacting physics

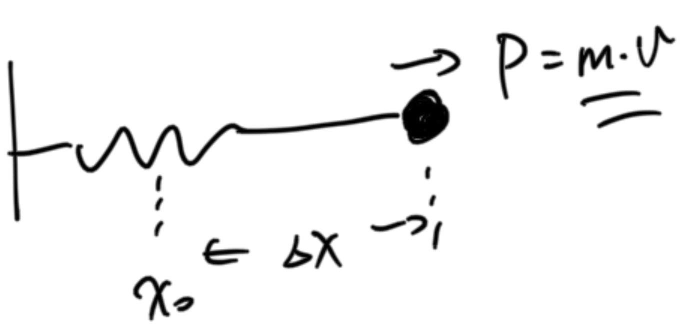
$[\hat{X}, \hat{X}^{+}] = 1$

$\hat{N} = \hat{X}^{+}\hat{X}$ → The Hilbert space is given by $|n\rangle$

$\hat{X}^{+} |n\rangle = \sqrt{n+1}\, |n+1\rangle$

$\hat{X} |0\rangle = 0$ ← $\hat{X} |n\rangle = \sqrt{n}\, |n-1\rangle$

- Harmonic oscillator



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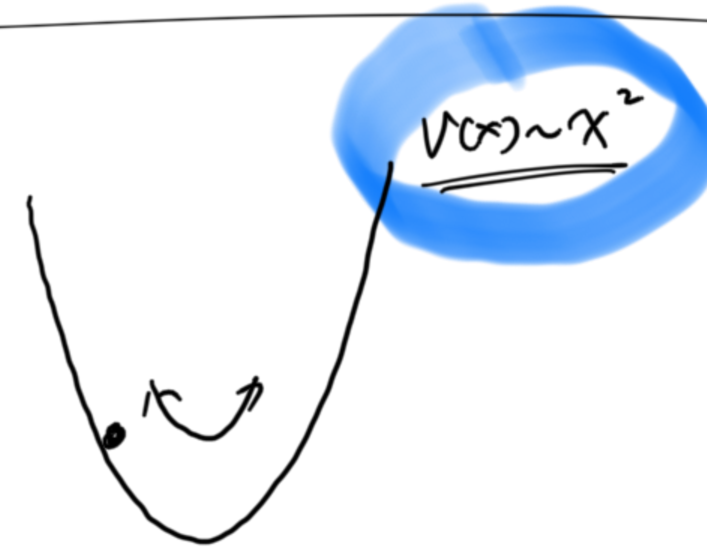


↓ also the motion of electrons in a magnetic field

$H = \frac{p^2}{2m} + \frac{1}{2} k \Delta x^2$

↓ spring constant

$\underline{\underline{\hat{A}}} = \hat{x}, \quad \underline{\underline{\hat{B}}} = -i\partial_x, \quad P = \hbar B = -i \underline{\underline{\hbar}} \partial_x$

P: momentum; $\hbar$: experimentally determined

$[\underline{\underline{\hat{x}}}, \underline{\underline{\hat{p}}}] = i\underline{\underline{\hbar}}$ → angular momentum

- $\hat{a} = \frac{1}{\sqrt{2\hbar}}(\lambda\hat{x} + i\hat{p}\cdot(\lambda\hbar)^{-1})$
  $\hat{a}^+ = \frac{1}{\sqrt{2\hbar}}(\lambda\hat{x} - i\hat{p}(\lambda\hbar)^{-1})$

λ has the dimension of inverse length

$$\boxed{[\hat{a}, \hat{a}^+] = 1 \qquad \begin{aligned} \hat{a}^+ a|n\rangle &= n|n\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n\rangle \\ \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}}$$

- The coherent state:

$$\hat{a}|0\rangle = 0 \rightarrow (\lambda\hat{x} + i\hat{p}\lambda^{-1}\hbar^{-1})|0\rangle = 0$$

$$\hat{p} = -i\hbar\partial_x$$

$$\Rightarrow \underline{\underline{\psi_0(x)}} = \underline{\underline{\langle x|0\rangle}} = \underline{\underline{\sqrt{\frac{\lambda}{\pi^{\frac{1}{2}}}}\, e^{-\frac{1}{2}x^2\lambda^2}}}$$

$$\Rightarrow \tilde{\psi}_0(p) = \langle p|0\rangle = \int dx\, \underbrace{\langle p|x\rangle}_{\sim e^{ipx}} \underbrace{\langle x|0\rangle}_{\psi_0(x)}$$

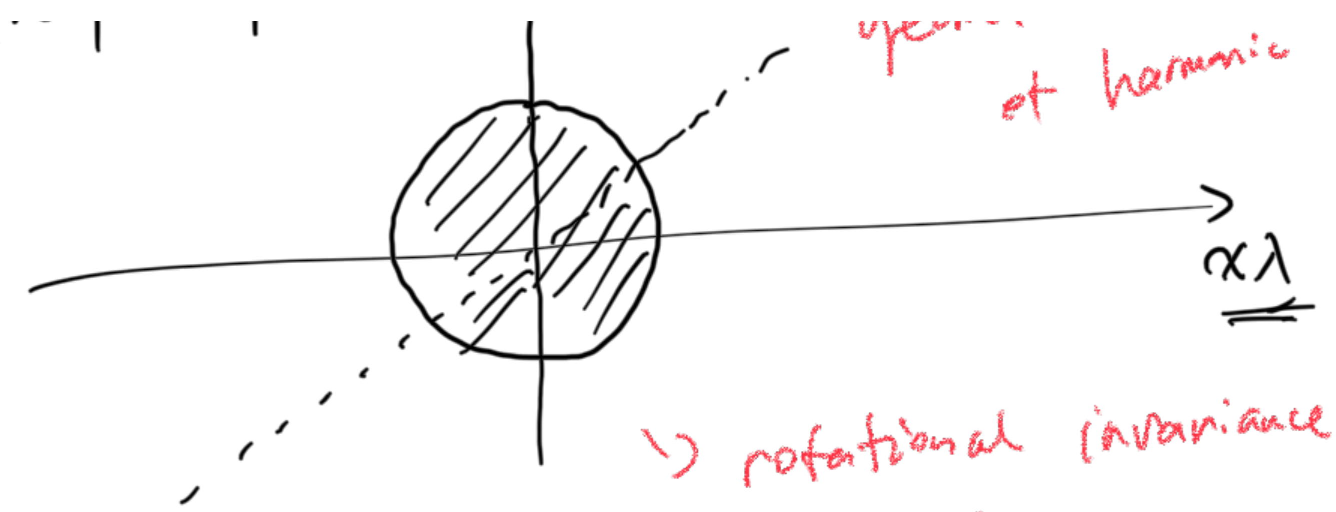
$$= \sqrt{\frac{1}{\lambda\pi^{\frac{1}{2}}}}\, e^{-\frac{1}{2}\underline{\underline{p^2\lambda^{-2}}}}$$

$$\begin{aligned} x' &= x\cos\theta + p\sin\theta \\ p' &= x\sin\theta - p\cos\theta \end{aligned} \Rightarrow \begin{aligned} \langle x'|0\rangle &\sim e^{-\frac{1}{2}\underline{\underline{x'^2\lambda^2}}} \\ \langle p'|0\rangle &\sim e^{-\frac{1}{2}\underline{\underline{p'^2\lambda^{-2}}}} \end{aligned}$$

(x', p') is a rotation (x, p)

the phase space $\quad \uparrow \underline{\underline{p\lambda^{-1}}}$

geometric aspect oscillator



of harmonic

$\rightarrow$ rotational invariance in phase space

- Bogoliubov transformation

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} + i\hat{p})$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar}}(\hat{x} - i\hat{p})$$

set $\lambda = 1$

$$\Rightarrow \text{☆} \begin{cases} \hat{b} = \cosh\theta\, \hat{a} + e^{i\phi}\sinh\theta\, \hat{a}^+ = \frac{1}{\sqrt{2\hbar}}(\hat{x}' + i\hat{p}') \\ \hat{b}^+ = \cosh\theta\, \hat{a}^+ + e^{-i\phi}\sinh\theta\, \hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x}' - i\hat{p}') \end{cases}$$

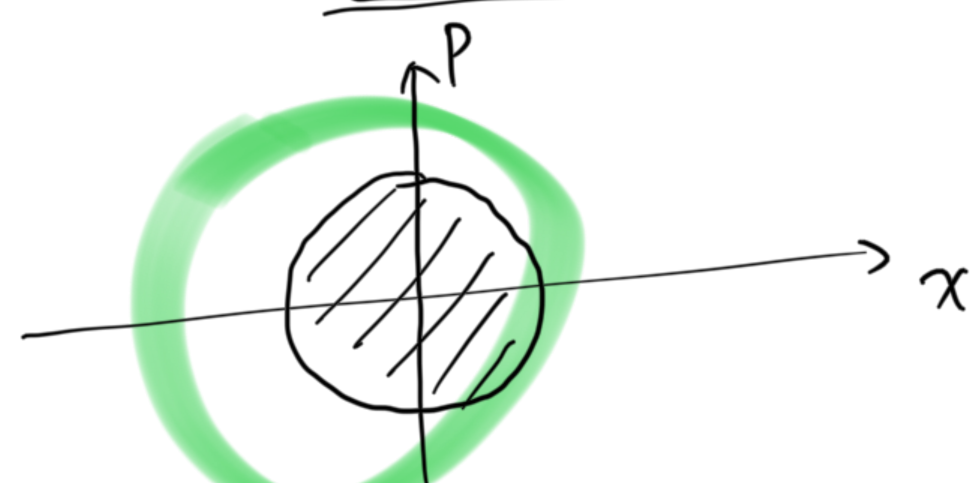
$$[\hat{b}, \hat{b}^+] = 1 = \cosh^2\theta - \sinh^2\theta$$

$$\boxed{\begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} = \boxed{\begin{pmatrix} \cosh\theta + \sinh\theta\cos\phi & \sinh\theta\sin\phi \\ \sinh\theta\sin\phi & \cosh\theta - \sinh\theta\cos\phi \end{pmatrix}} \text{☆} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}}$$

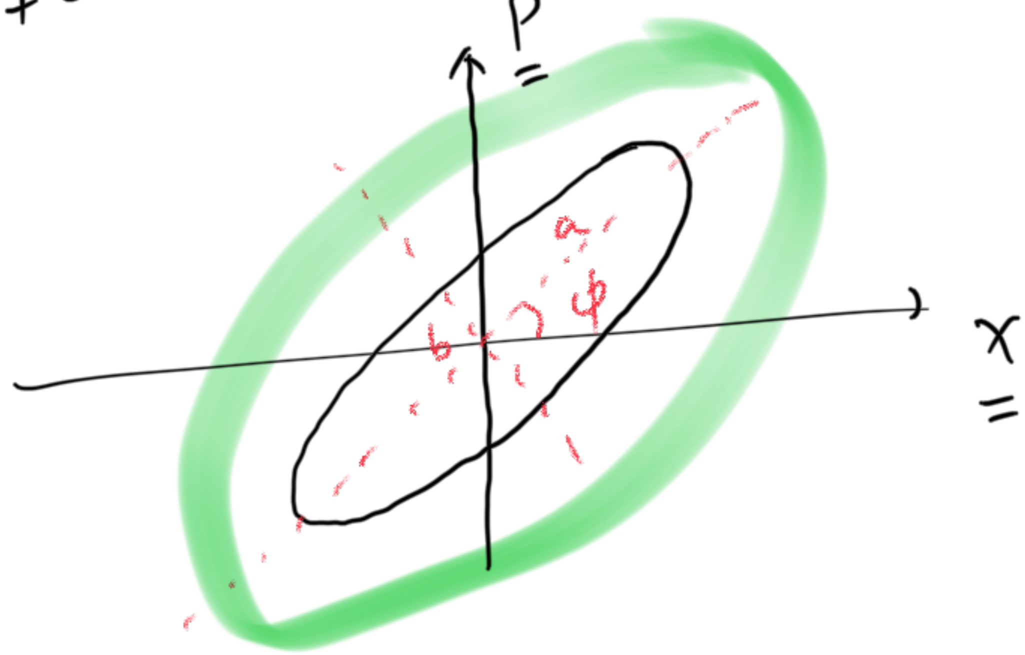
$\Downarrow$

a unimodular metric g with $\det(g) = 1$

If $\theta = 0$, $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



If $\theta \neq 0$



$\frac{a}{b} \sim \cosh\theta$

For a dynamical system, $\underline{\lambda}$, θ, ϕ are fixed by Hamiltonians

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\underbrace{m\omega^2}_{k} x^2$$

$$\underline{\underline{\lambda = \sqrt{m\omega}}}, \quad \theta = 0$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\cdot\hat{x} + \frac{i\hat{p}}{\sqrt{m\omega}}\right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{x} - \frac{i\hat{p}}{\sqrt{m\omega}}\right)$$

$$\hat{H} = (\hat{x}, \hat{p})\begin{pmatrix} \frac{1}{2}m^2\omega^2 & 0 \\ 0 & \frac{1}{2m} \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}$$

$$\hat{H}' = (\underline{\underline{\hat{x}'}}, \underline{\underline{\hat{p}'}})\begin{pmatrix} \frac{1}{2}m\omega^2 & 0 \\ 0 & \frac{1}{2m} \end{pmatrix}\begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix}$$

$$= \left(\frac{1}{2m}\sinh^2\theta\sin^2\phi + \frac{1}{2}m^2\omega^2(\cosh\theta + \sinh\theta\cos\phi)^2\right)\underline{\hat{x}^2}$$

$$+ \left(\frac{1}{2m}(\cosh\theta - \sinh\theta\cos\phi)^2 + \frac{1}{2}m^2\omega^2\sin^2\phi\sinh^2\theta\right)\underline{\hat{p}^2}$$

$$+\left(\frac{1}{2m}\sin\phi\sinh\theta\,(\cosh\theta-\sinh\theta\cos\phi)+\frac{1}{2}m^2\omega^2\sin\phi\sinh\theta\,(\cosh\theta+\sinh\theta\cos\phi)\right)(\hat{x}'\hat{p}+\hat{p}\hat{x})$$

$$=\frac{\hat{p}'^2}{2m}+\frac{1}{2}m^2\omega^2\hat{x}'^2$$

$\hat{D}=\hat{x}\cdot\hat{p}\sim -i\hbar x\cdot\partial_x \rightarrow$ Dilation generator

$\hat{U}_c=e^{-ic\cdot\hat{D}} \rightarrow$ dilation operator

$$f(x)\rightarrow \quad x\rightarrow(1+\epsilon)\cdot x$$

$$\lim_{\epsilon\to 0} f((1+\epsilon)x)=f(x)+\epsilon\cdot x\partial_x f(x) \quad \|$$

$$=\hat{U}_\epsilon f(x)$$

· 2D surface of a sphere (Free particles)

· Angular momentum operators ($\hat{L}=\hat{r}\times\hat{p}$)

$$\hat{L}^a=\epsilon^{abc}g_{bd}\hat{r}^d\hat{p}_c$$

$$[\hat{L}^a,\hat{L}^b]=i\epsilon^{abc}\hbar\hat{L}^c$$

· Kinetic energy of a particle on the sphere

· $\hat{H}=\dfrac{\hat{p}_x^2+\hat{p}_y^2}{2m}\rightarrow$ on a plane

$\hat{L}^2$ total

$I = m \cdot R^2$

$\cdot\ \hat{H} = \dfrac{\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}{2I}$ → angular momentum

$2I$ → moment of inertia

→ on the sphere

$$= \frac{\hat{L}^2}{2I}$$

Since $[\hat{L}^2, \hat{L}_z] = 0$, the two operators share the same eigenstates

$$\hat{L}^2 |m, l\rangle = l\cdot(l+1)\,|m, l\rangle$$

$$\hat{L}_z |m, l\rangle = m|m, l\rangle$$

· Kinetic energy

On the plane, $\underline{\underline{E}} = \dfrac{k_x^2 + k_y^2}{2m}\cdot \hbar^2$

two quantum numbers $\underline{\underline{k_x}},\ \underline{\underline{k_y}}$

the eigenstates $\underline{e^{ik_x x} e^{ik_y y} = \varphi_{k_x, k_y}(x,y)}$

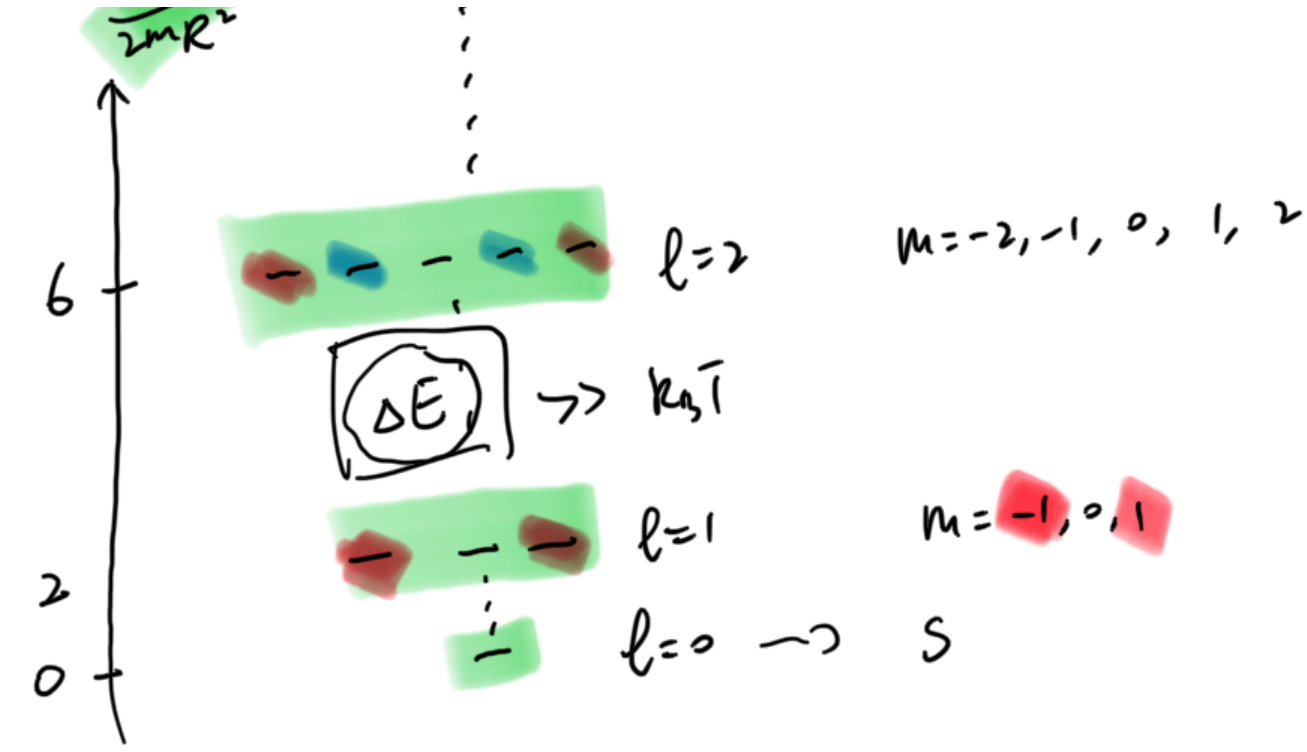
On the sphere $E = \dfrac{l\cdot(l+1)\hbar^2}{2I}$

two quantum numbers l, m

The eigenstates are

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}\; e^{i\underline{m}\phi}\;\frac{1}{\sin^m\theta}\cdot\frac{d^{l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l}$$

E



- Unlike the plane, kinetic energy levels on the sphere are discrete for a finite radius R

  (sphere is compact, plane is not)

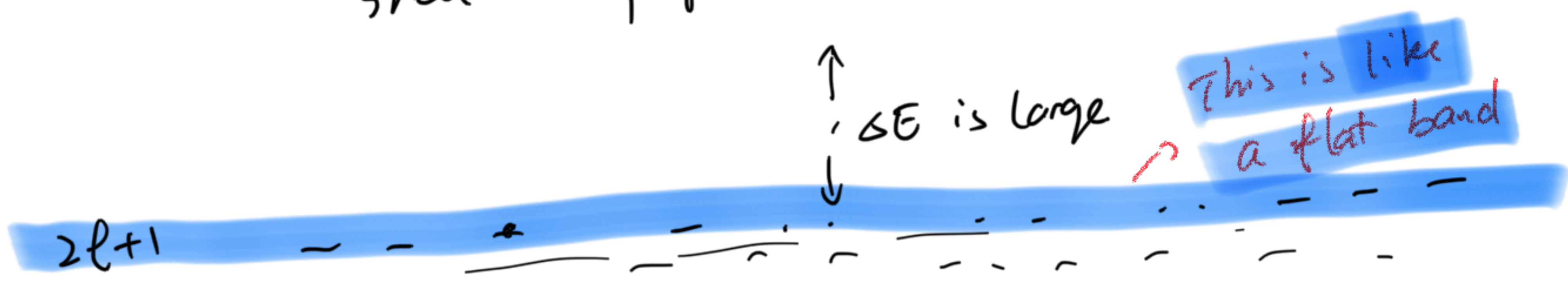
- The "Landau level" like structure of energy spectrum is due to the geometric <u>curvature</u> of the sphere, mimicking an "effective magnetic field"

- The interacting Hamiltonian

$$\Delta E_\ell = \frac{\hbar^2}{2I}\left((\ell+1)(\ell+2) - \ell\cdot(\ell+1) \right)$$

$$= \frac{\hbar^2}{I}\,\underline{(\ell+1)} \quad ||$$

For large ℓ, ΔE is large, only top most shell is physically relevant



$=$

$|m, l\rangle, \quad m = -l, -l+1, \ldots l-1, l$

$$\hat{H} = \int d^2r_1 \, d^2r_2 \, \frac{1}{|r_1 - r_2|} \, \rho_{r_1} \, \rho_{r_2}$$