Lecture 2: Integer Quantum Hall Effect

- Two-dimensional system

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2\right)$$

$$0 \le x \le L_x$$
$$0 \le y \le L_y \qquad V = L_x L_y L_z$$
$$0 \le z \le L_z$$

$$\psi_{n_1 n_2 n_3}(x,y,z) = \sqrt{\frac{8}{V}} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \sin\left(\frac{n_3 \pi z}{L_z}\right)$$

$$E_{n_1 n_2 n_3} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_1^2}{L_x^2} + \frac{n_2^2}{L_y^2} + \frac{n_3^2}{L_z^2}\right)$$

Let $L_z \ll L_x$, $L_z \ll L_y$

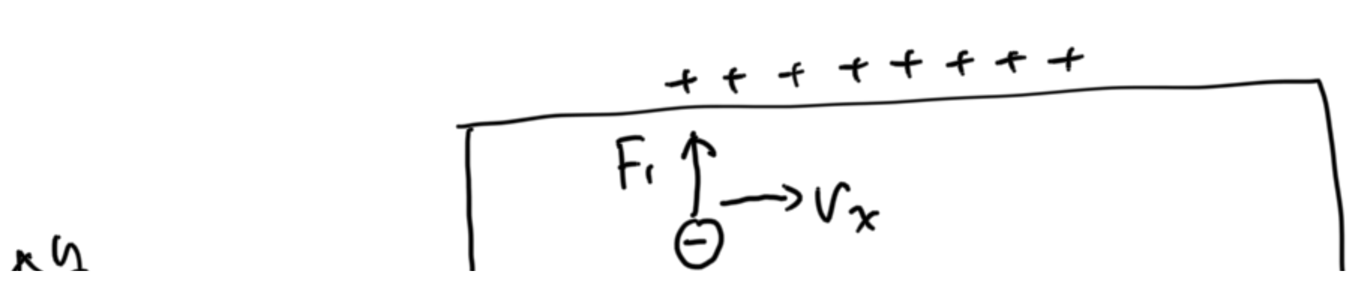
$$\Delta E_z = \frac{3 \cdot \pi^2 \hbar^2}{2m L_z^2} \gg k_B T \Rightarrow \text{2D system}$$

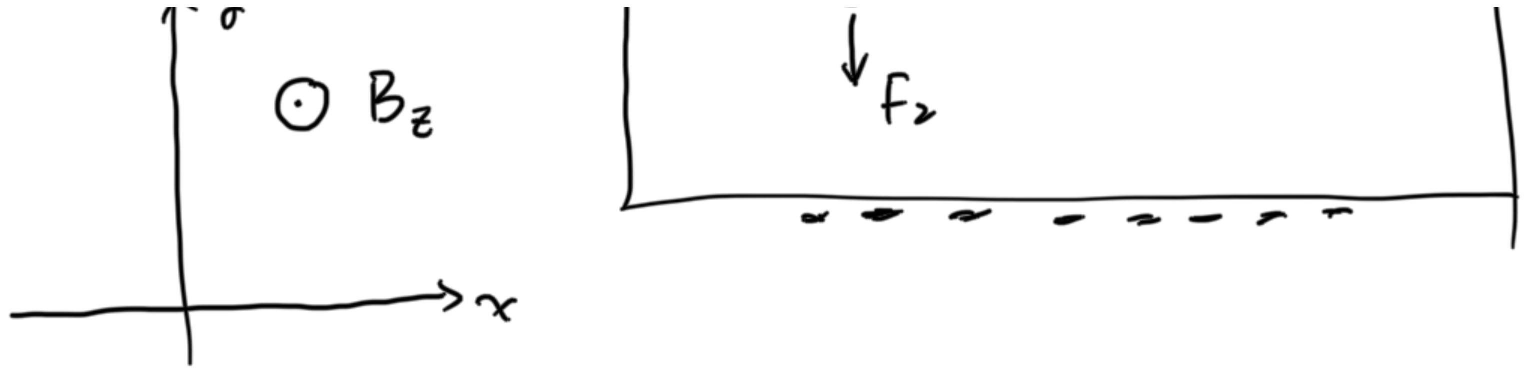
(no degrees of freedom along z-direction)

Similar arguments for 1D and 0D systems

- Dimensionality depends on temperature

- 2D Hall effect





$F_1 = E_y \cdot e$

$F_2 = B_z \cdot e \cdot V_x$

$\Rightarrow \quad F_1 = F_2 \rightarrow V_x = \dfrac{E_y}{B_z}$

current density

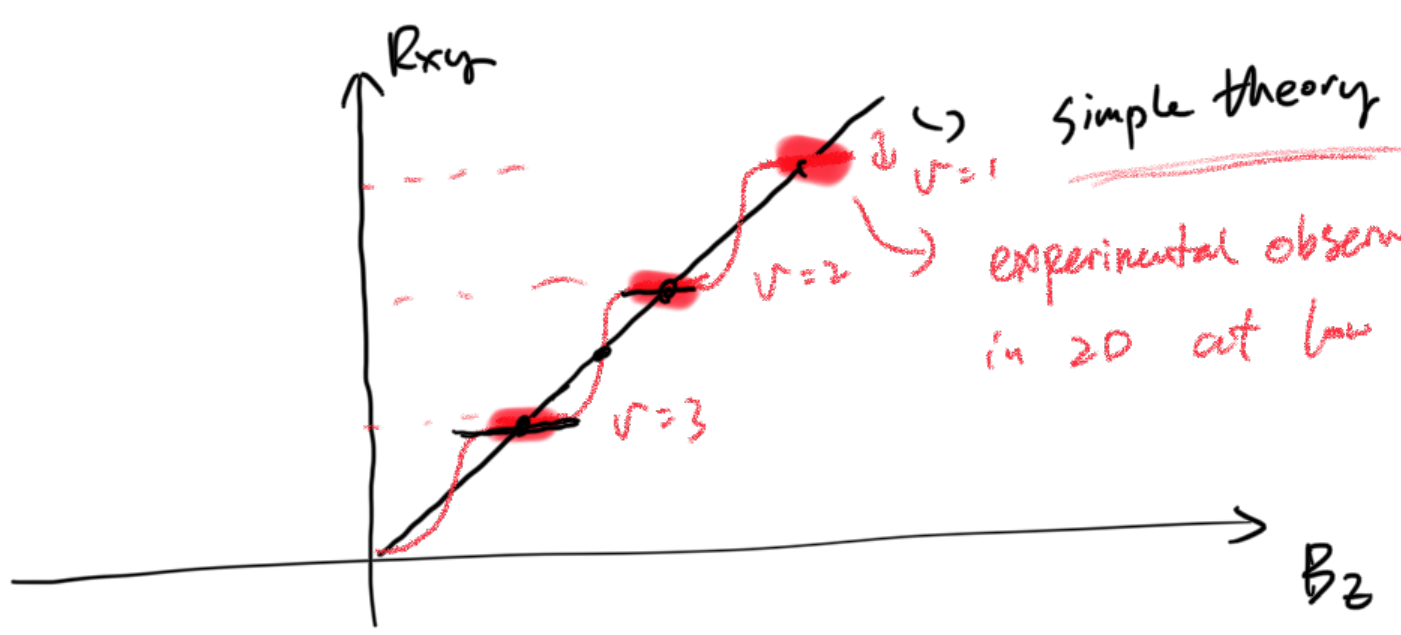
$J_x = en \cdot v_x = \dfrac{enE_y}{B_z}$

number density

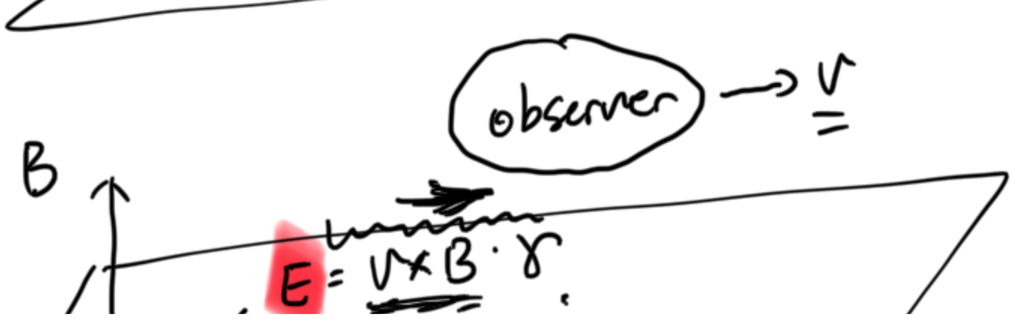
$R_H = \dfrac{E_y}{J_x \cdot B_z} = \dfrac{1}{en}$ → charge density

Hall coefficient

$J_x = E_y \cdot \dfrac{R_H}{B_z} \rightarrow \sigma_{xy} = \dfrac{R_H}{B_z}, \quad R_{xy} = \dfrac{B_z}{R_H}$



· Hall effect from Lorentz invariance



$\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1$

$=$

$e = 1$

$J = n \cdot v$

$$\frac{E}{J \cdot B} = \frac{1}{n} = R_H$$

- Observation of Hall plateau

  - Ground state gap ✡ → Larger than temperature and disorder

  - Anderson localization in the bulk ✓
    due to disorder, breaks Lorentz invariance

  - Topological "obstruction" of Anderson localization at the edge ✓

only important for plateau to be seen in transport measurement

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- A formal treatment

  - Phase space of an electron in 2D

    $\hat{r}^x, \hat{r}^y, \hat{P}_x, \hat{P}_y,\quad [\hat{r}^a, \hat{P}_b] = i\delta^a_b$

  - Without a B field

    $$\hat{H} = \frac{1}{2m}(\hat{P}_x^2 + \hat{P}_y^2)$$

  - By applying a magnetic field (minimal couplings)

    $$\hat{P}_a \to \hat{P}_a - eA_a$$

    → vector potential

U(1) gauge theory

$\psi(x) \to e^{i\alpha}\psi(x)$

$\to e^{i\alpha(x)}\psi(x)$

$\alpha(x) \sim \int dl \cdot \vec{A}$

$$\hat{H} = \frac{1}{2m}\left((\hat{P}_x - eA_x)^2 + (\hat{P}_y - eA_y)^2 \right)$$

$$B_z = \nabla \times \vec{A} = \partial_x A_y - \partial_y A_x$$

magnetic length $\ell_B = \sqrt{\frac{\hbar}{eB}}$ (set $\hbar = 1$)

. Two sets of spatial coordinates

$\epsilon^{xy} = 1$
$\epsilon^{yx} = -1$

cyclotron coordinates $\leftarrow$ $\tilde{R}^a = -\epsilon^{ab}\ell_B^2(\hat{P}_b - eA_b)$

$$[\tilde{R}^a, \tilde{R}^b] = i\,\ell_B^2\,\epsilon^{ab}$$

guiding center coordinates $\leftarrow$ $\bar{R}^a = \hat{r}^a - \tilde{R}^a$ $\qquad (\bar{R}^a + \tilde{R}^a = \hat{r}^a)$

$$[\bar{R}^a, \bar{R}^b] = -i\,\ell_B^2\,\epsilon^{ab}$$ $\hookrightarrow$ uncertainty principle

$$[\tilde{R}^a, \bar{R}^b] = 0$$

. The kinetic energy

$\hat{H} = \hat{H}(\tilde{R}^a)$ $\to$ general for any system

$$[\tilde{R}^a, \tilde{R}^b] = i\,\epsilon^{ab}\ell_B^2$$

$\to$ Harmonic oscillator

The "$\hbar$" of this 2D system is ℓ_B^2, tunable with the magnetic field

$\to$ $\hat{a} = \frac{\ell_B^{-1}}{\sqrt{2}}(\tilde{R}^x + i\tilde{R}^y)$ $\qquad [\hat{a}, \hat{a}^\dagger] = 1$

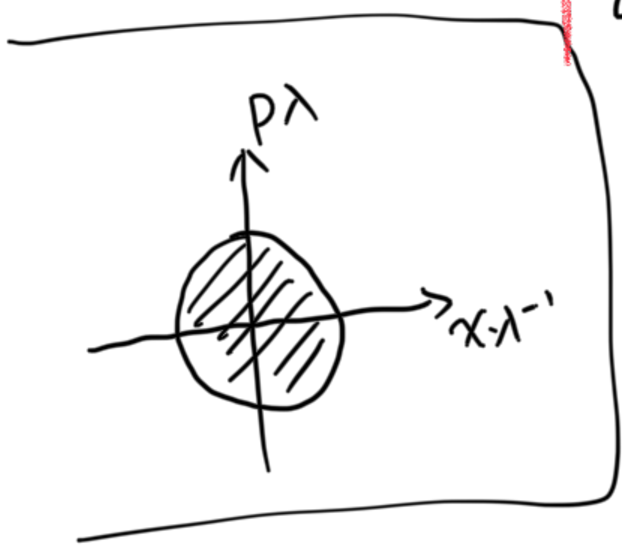
$$\hat{a}^{+} = \frac{\ell_B^{-1}}{\sqrt{2}} (\tilde{R}^x - i\tilde{R}^y)$$

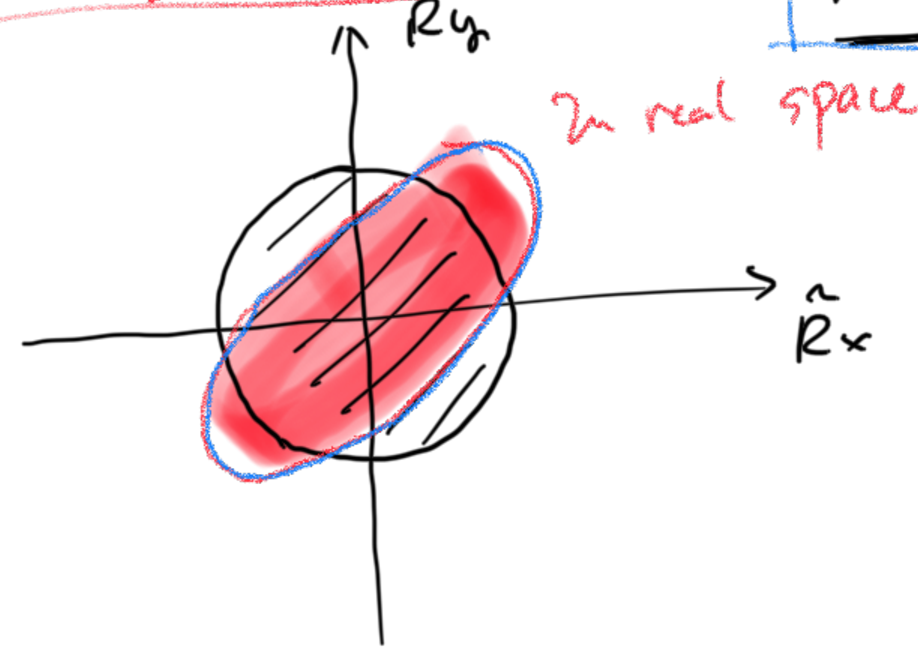
$|m\rangle$

$$\rightarrow \boxed{\begin{aligned} \tilde{a}_g &= \cosh\theta\, \hat{a} + \sinh\theta\, e^{i\phi} \hat{a}^{+} \\ \tilde{a}_g^{+} &= \cosh\theta\, \hat{a}^{+} + \sinh\theta\, e^{-i\phi} \hat{a} \end{aligned}} \Rightarrow g(\theta, \phi)$$

$$\boxed{|n\rangle_g = \sum_m |m\rangle \cdot C_m}$$

Landau level mixing





phase space $\longrightarrow$ real space

$\hbar \longrightarrow \ell_B^2$

$$\rightarrow \tilde{a}_g |\tilde{0}\rangle_g = 0 \qquad \tilde{a}_g |n\rangle_g = \sqrt{n}\, |n-1\rangle_g$$

$$\tilde{a}_g^{+} |n\rangle_g = \sqrt{n+1}\, |n+1\rangle_g$$

$|n\rangle_g \rightarrow$ Landau levels

· The simplest example: (Galilean invariant system)

$$\hat{H} \sim |\hat{P}|^2 \Leftarrow \hat{H}_G = \frac{1}{2m}\left((\hat{P}_x - eA_x)^2 + (\hat{P}_y - eA_y)^2 \right)$$

$$= \frac{1}{2m\ell_B^4} \left(\tilde{R}^{x\,2} + \tilde{R}^{y\,2} \right)$$

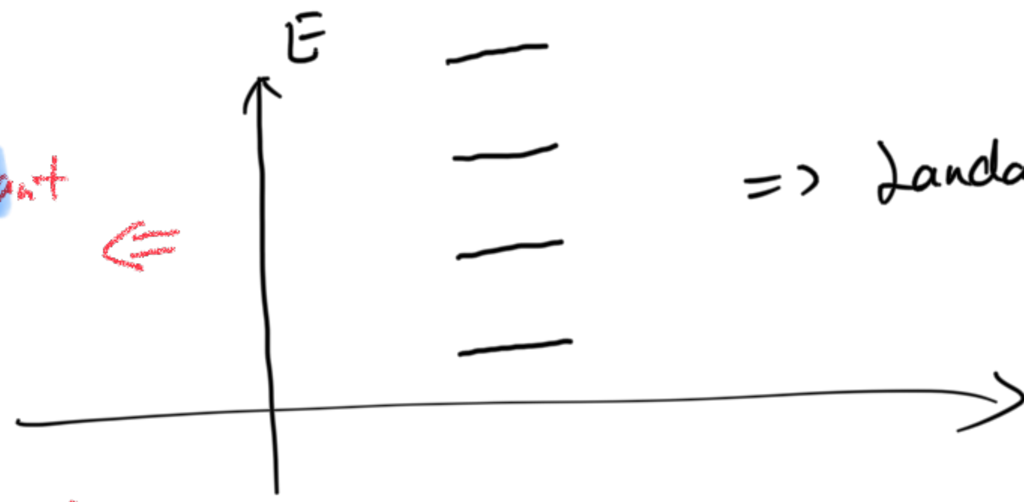
$$= \frac{1}{2m\ell_B^2} \left(a_\eta^{+} a_\eta + a_\eta a_\eta^{+} \right)$$

$$g(\theta = 0) = \eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{H}|n\rangle_\eta = \frac{1}{m\ell_B^2}\left(n+\frac{1}{2}\right)|n\rangle$$

$$= \hbar w_c \left(n+\frac{1}{2}\right)|n\rangle$$

$w_c = \dfrac{eB}{m}$ is the cyclotron energy



$\Rightarrow$ Landau levels are equally spaced

Galilean invariant system, $\Leftarrow$

$E \sim p^2$

The equations of motion from $\mathcal{L} \sim \dot{x}^2$ is invariant when $\dot{x} \to \dot{x} + v$

$\mathcal{L} = \dot{x}^2 \frac{m}{2} - V(x)$

$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \to$ e.o.m

$\dot{x} \to \dot{x} + v_c$

$\frac{\partial V}{\partial x} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(\dot{x}+v_c)^2\right) = \frac{d}{dt}\left(\frac{\partial \dot{x}^2}{\partial \dot{x}}\right)$

· Anisotropic generalization

effective mass tensor

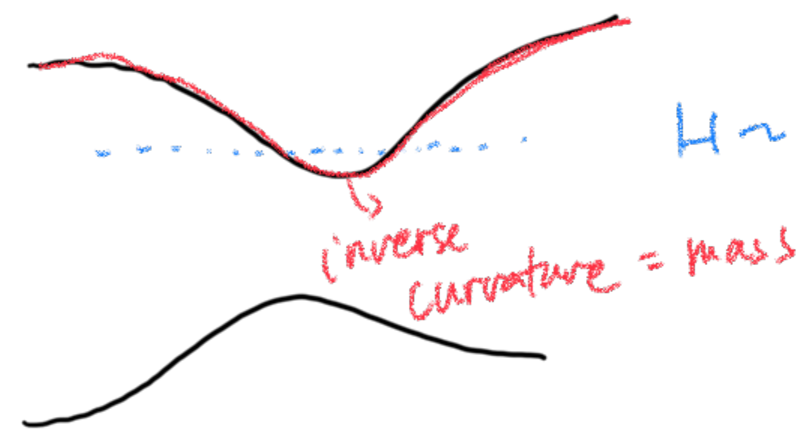
$\sim \hat{H} = \frac{1}{2m\ell_B^2} g_{ab} \tilde{R}^a \tilde{R}^b \equiv \hat{H}_G$ when $\theta = 0$

$$g = \begin{pmatrix} \cosh\theta + \sinh\theta \sin\phi, & \sinh\theta\cos\phi \\ \sinh\theta\cos\phi & \cosh\theta - \sinh\theta\sin\phi \end{pmatrix}$$

$$\hat{H} = \frac{1}{2m\ell_B^2}\left(\hat{a}_g^\dagger \hat{a}_g + \hat{a}_g \hat{a}_g^\dagger\right)$$

· The Landau level wavefunctions will be squeezed

- The LL energy is still equally spaced



$H \sim |\underline{\underline{k}}|^2 \sim \underline{\underline{g^{ab}}} k_a k_b$

– Going beyond Galilean invariance

$$\hat{H} \sim \underset{=}{}k^2 + \underset{=}{}k^4 + \underset{=}{}k^6 \; ---$$

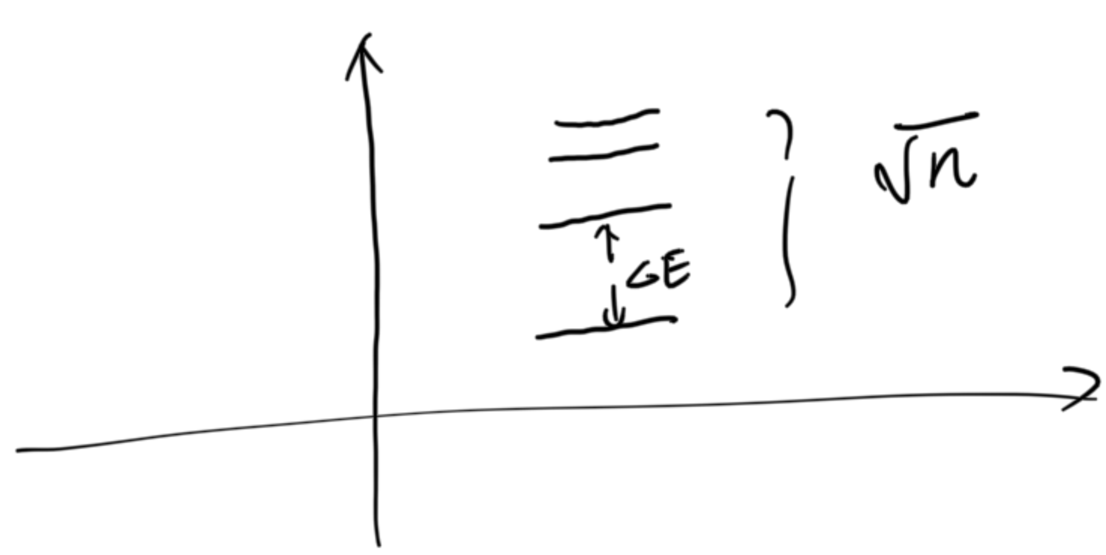
After minimal coupling

$$\hat{H} \sim |\tilde{R}|^2 + |\tilde{R}|^4 + |\tilde{R}|^6$$

$$\sim \underline{\underline{\hat{a}^\dagger \hat{a}}} + \underline{\underline{(\hat{a}^\dagger \hat{a})^2 + (\hat{a}^\dagger \hat{a})^3}} \cdots$$

For graphene

$$\hat{H} \sim v_F \cdot |k| \sim \sqrt{\hat{a}^\dagger \hat{a}}$$



- $[\underline{\bar{R}}^a, \underline{\underline{\hat{H}(\underline{\tilde{R}}^a)}}] \underline{\underline{= 0}} \rightarrow$ <u>degeneracy</u> within each LL.

$$\hat{b} = \frac{\ell_B^{-1}}{\sqrt{2}} (R^x - i R^y) \qquad [\hat{b}, \hat{b}^\dagger] = 1$$

$$\hat{b}^\dagger = \frac{\ell_B^{-2}}{\sqrt{2}} (\bar{R}^x + i \bar{R}^y)$$

$$\underline{\underline{\hat{b}}}\ \underline{\underline{|n\rangle}} \rightarrow \underline{\underline{|n'\rangle}}$$

- The amount of degeneracy

$$N_0 = \frac{A}{2\pi \ell_B^2}$$

A → area of the sample

$2\pi \ell_B^2$ ⇒ unit area of a single magnetic flux

$$\underline{\underline{[\bar{R}^a, \bar{R}^b] = -i \epsilon^{ab} \ell_B^2}}$$

- The filling factor

$$\nu = \frac{N_e}{N_0}$$

If ν is an integer, we have fully filled LL at $T=0$

↓

flat band

⇒ integer quantum Hall effect

⇒ topological insulator, each LL has $C = 1$ without any symmetry protection

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- QHE on spherical geometry

For $B = 0$

$$H = \frac{1}{2I} \hat{L}^2$$

$$\hat{L} = \begin{pmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{pmatrix} = \vec{r} \times \underline{\underline{\vec{p}}}$$

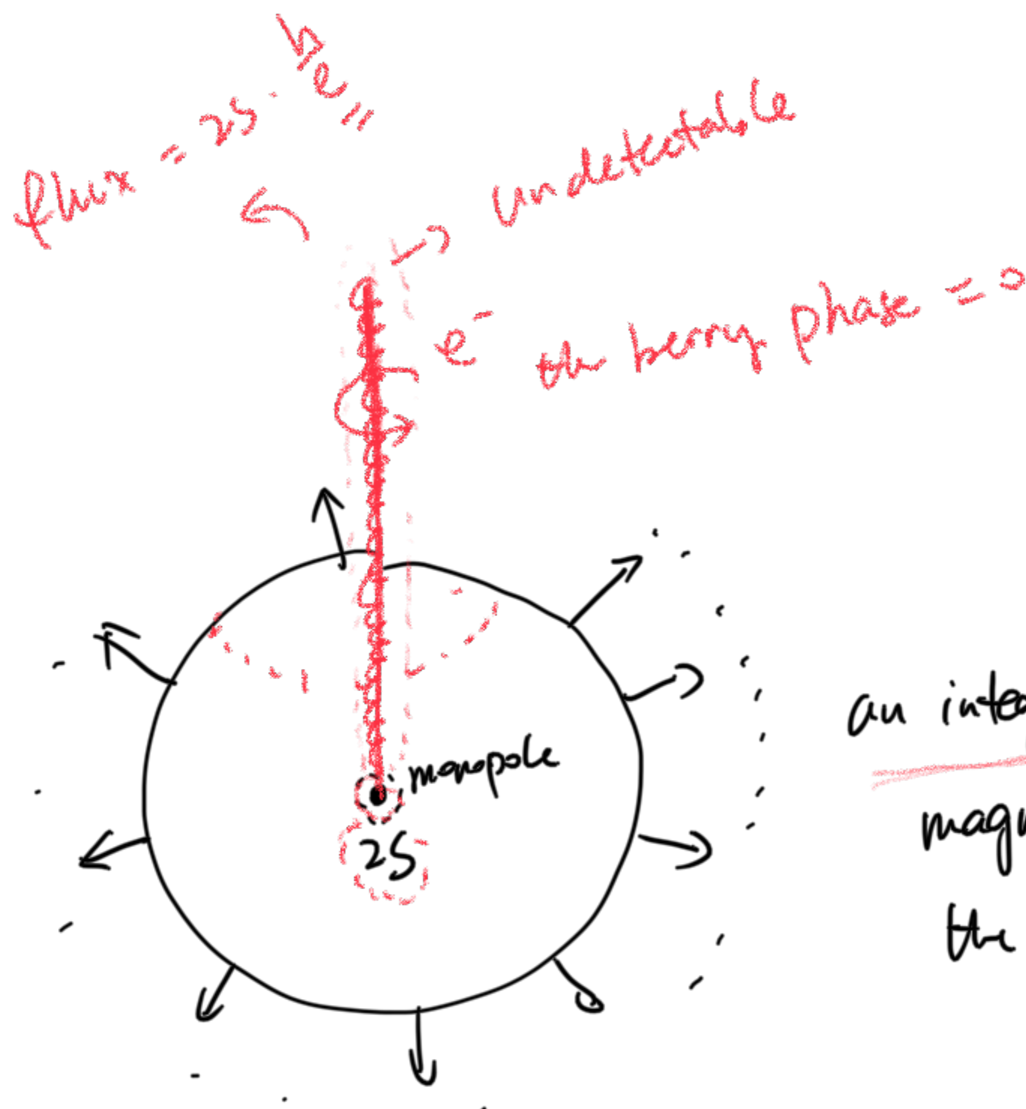
$(\quad L_z \quad)$

For $B > 0$

$$H = \frac{1}{2I}\underline{\underline{|\hat{\Lambda}|^2}} = \underline{\underline{\frac{1}{2}\omega_c \cdot \frac{|\hat{\Lambda}|^2}{\hbar \cdot S}}} = \underline{\underline{\frac{1}{2}\frac{\omega_c}{\hbar S}(\hat{L}^2 - \hbar^2 S^2)}}$$

$$\hat{\Lambda} = \hat{r} \times (\underline{\hat{p} - e\hat{A}})$$

$$\underline{\underline{\vec{\nabla} \times A = B \cdot \hat{\Omega}}}$$



$(2S)$
an integer number of magnetic flux through the sphere

The eigenstates are given by $|\ell\rangle$

with $$\hat{H}|\ell\rangle = \frac{1}{2}\frac{\omega_c}{\hbar S}\left(\underline{\ell(\ell+1)\hbar^2 - \hbar^2 S^2}\right)|\ell\rangle$$

$\ell \geq S, \quad \ell = S, S+1, \ldots S+n$

n: Landau level index

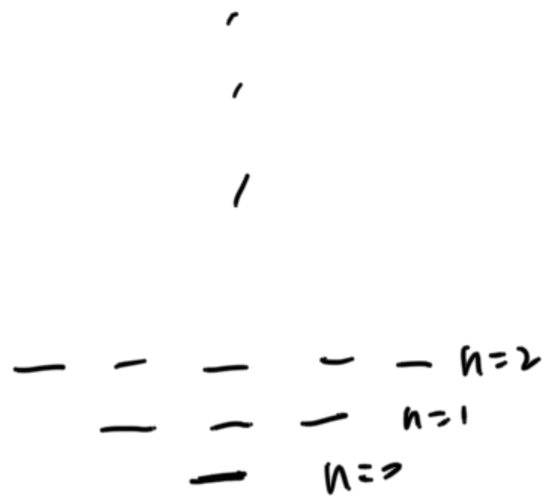
$$\boxed{L_z|m,\ell\rangle = m|m,\ell\rangle}$$

$$|m,\ell\rangle = \underline{\underline{|\ell\rangle}} = |S+n\rangle = |n\rangle$$

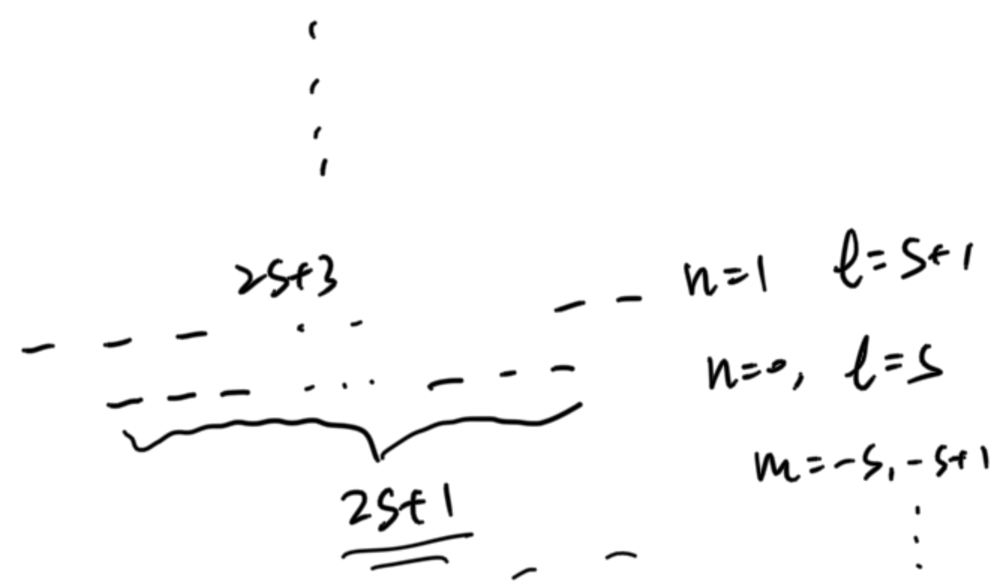
Thus $$\hat{H}|n\rangle = \frac{1}{2}\hbar\frac{\omega_c}{S}\left(\underline{n(n+1) + (2n+1)S}\right)|n\rangle$$

$S=0$ $(B=0)$

$S>0$ $(B>0)$



$$\Delta E = \frac{(n+1)\hbar^2}{mR^2}$$



$$\Delta E = \hbar\omega_c + \frac{(n+1)\cdot\hbar^2}{mR^2}$$

R is the radius of the sphere

$R \to \infty$ goes back to the plane

- Degeneracy within a single LL on the sphere ✓✓

	On the sphere	On the plane
Number of orbitals	$N_0 = 2S+1 + 2n$ (LL index)	$N_0 = \frac{A}{2\pi l_B^2}$
$A = 4\pi R^2$	$= \frac{A}{2\pi l_B^2} + 2n+1$	
Number of electrons for $\nu = 1$	$N_e = N_0$	$N_e = N_0$
N_{qh} "quasi" hole	$N_e = N_0 - N_{qh}$	$N_e = N_0 - N_{qh}$

For ground state 1 1 1 1 1 .. 1 1 .. 1 1 $\quad N_\phi = N_e, \quad L_z = L = 0$

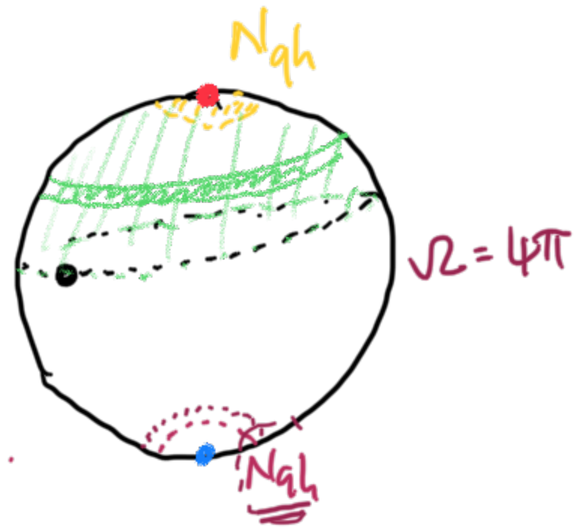
For N_{qh} quasiholes at the north pole: $\underbrace{\overset{S\ S-1}{0\ 0} \ldots 0}_{N_{qh}}\ 1\ 1\ 1 \ldots \overset{-S}{1\ 1}$ $\quad N_e = N_\phi - N_{qh}$

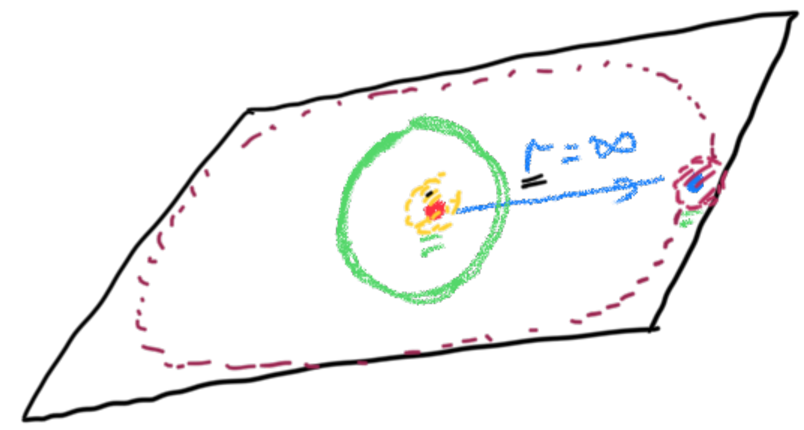
$L_z = -\frac{N_e}{2} \cdot N_{qh}$

$\overset{\frac{5}{2}}{1}\ 1\ 1\ 1\ 1\ \overset{-\frac{5}{2}}{1}$ $\quad L_z = 0$

$0\ 1\ 1\ 1\ 1\ 1$ $\quad 2S+1 = 6 \quad S = \frac{5}{2} \quad L_z = -\frac{5}{2}$

$\frac{5}{2} \quad \frac{3}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{3}{2} \quad -\frac{5}{2}$





$L_z \left| \psi^{N.P}_{N_{qh}} \right\rangle = -\frac{N_e}{2} \cdot N_{qh} \left| \psi^{N.P}_{N_{qh}} \right\rangle$ $\qquad \gamma = 0$

$\gamma = -2\pi \cdot \frac{N_e}{2} \cdot N_{qh}$

$L_z \left| \psi^{S.P}_{N_{qh}} \right\rangle = \frac{N_e}{2} \cdot N_{qh} \left| \psi^{S.P}_{N_{qh}} \right\rangle$ $\qquad \gamma' = 2\pi \cdot N_e \cdot N_{qh}$

$\gamma' = 2\pi \cdot \frac{N_e}{2} \cdot N_{qh}$

The Berry phase of rotation or adiabatically dragging N_{qh} holes in a loop on the sphere

$\gamma = \frac{N_e}{2} \cdot N_{qh} \cdot \Omega$ $\qquad \gamma_0 = 2\pi \cdot \phi \cdot N_{qh}$

$= \frac{1}{2}(N_\phi - N_{qh}) \cdot N_{qh} \cdot \Omega$

$= \frac{1}{2}\left(\frac{A}{2\pi l_B^2} + 2n + 1 - N_{qh}\right) N_{qh} \cdot \Omega$

γ_0 ϕ

$= \frac{1}{2} N_{qh} \cdot (2S)$ → contribution from the magnetic field.

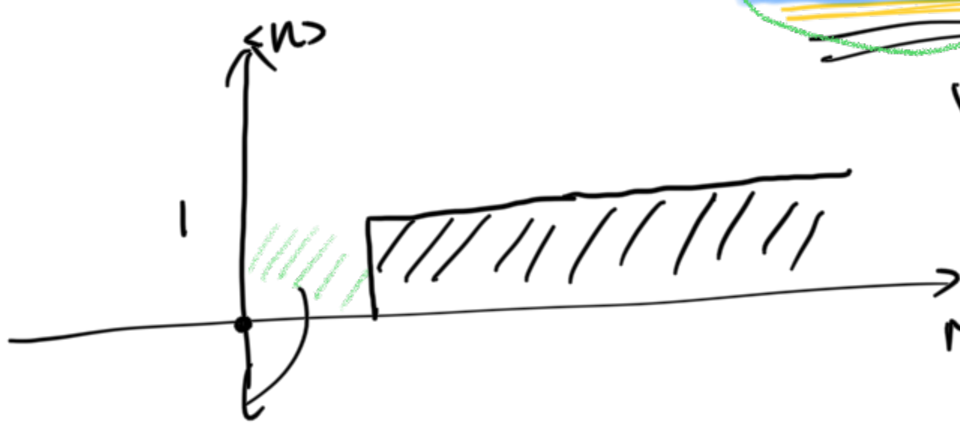
$- \frac{1}{2} N_{qh}^2 \Omega$ → orbital angular momentum contribution

$+ \frac{1}{2}(2n+1) \cdot N_{qh} \cdot \Omega$ → intrinsic topological spin contribution

Coupling of spin/angular momentum to the geometric curvature of the sphere

→ the topological spin $= n + \frac{1}{2} = s$

→ is this an independent topological index?



N_{qh} quasiholes

$$L_z = \frac{1}{2} + \frac{3}{2} + \cdots \frac{2N_{qh}-1}{2}$$

$$= \frac{1}{2} N_{qh}^2$$

An adiabatic exchange gives a Berry phase $e^{i \cdot s \cdot 2\pi} = -1$ for 2QH quasiholes

· This implies an 2QH quasihole is a fermion

The wavefunction of the two-hole state:

$z_i = x_i + i y_i$

$$\Psi_{a_1, a_2}(z_1, \cdots z_N)$$

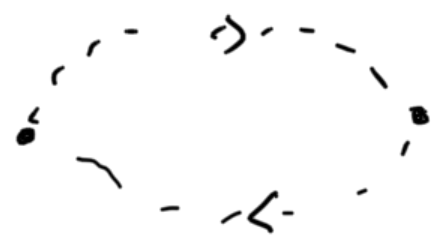
$$\sim \prod_{i,j} (z_i - a_1)(z_j - a_2) \cdot \prod_{i',j'} (z_{i'} - z_{j'}) \cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$$

... explicitly anti-symmetrized)

(a_1, a_2 are not explicitly [illegible])

- Two particles in a single LL

$$\psi(r_1, r_2) \sim \psi_1(r_1)\,\psi_2(r_2) \pm \psi_1(r_2)\,\psi_2(r_1)$$



An adiabatic exchange gives $e^{i\pi}$ for fermions
$e^{-i\pi}$ for bosons

(The statistics is put in by hand)

- What happens when $a_1 = a_2$

$a_1 = a_2$