Lecture 4. Fractional Quantum Hall Effect.

- The microscopic Hamiltonian

$$\vec{r}^a = \underbrace{\bar{R}^a}_{\text{within a single LL}} + \underbrace{\tilde{R}^a}_{\text{between different LL}}$$

- Only depends on $\bar{R}^x, \bar{R}^y$

$$[\bar{R}^x, \bar{R}^y] = -i\ell_B^2$$

$$b = \frac{1}{\sqrt{2}}(\bar{R}^x - i\bar{R}^y), \quad b^\dagger = \frac{1}{\sqrt{2}}(\bar{R}^x + i\bar{R}^y)$$

$$[b, b^\dagger] = 1$$

Single particle basis is a single LL,

$$|m\rangle = \frac{1}{\sqrt{m!}}(b^\dagger)^m |0\rangle, \quad b|0\rangle = 0$$

- $$\hat{H} = \int d^2q\, V_q\, \bar{\rho}_q \bar{\rho}_{-q} + \delta H$$

δH → goes to zero when $B \to \infty$

→ 3 or more-body interaction → treat interaction perturbatively

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Schreiffer-Wolff transformation

$$\bar{\rho}_q = \sum_i e^{iq\bar{R}}$$

$$= \int d^2q\, V_q \sum_{i \neq j} e^{iq(\bar{R}_i - \bar{R}_j)}$$

$$+ \text{one-body} + \delta H$$

one-body → Constant for translationally invariant system

- For 3-body Hamiltonian

$$H_{3body} = \int d^2q_1 d^2q_2\, V_{q_1 q_2} \sum_{i \neq j \neq k} e^{iq_1\bar{R}_i} e^{iq_2\bar{R}_j} e^{-i(q_1+q_2)\bar{R}_k}$$

$$= \int d^2q_1 d^2q_2\, V_{q_1 q_2}\, \bar{\rho}_{q_1} \bar{\rho}_{q_2} \bar{\rho}_{-q_1-q_2}$$

- two-body - one-body

. The strongly correlated topological phases

Ground state properties

$$N_{orb}^{*} = \frac{p}{q} N_e \quad \left(\frac{A}{2\pi l_B^2}\right)$$

(torus) → genus-1
→ no curvature

$$N_{orb}^{*} = \frac{p}{q}(N_e + S_e) - S_h$$

no. of orbitals; no. of electrons

(sphere, disk, cylinder)

genus-zero
→ non-trivial curvature

$\nu = \frac{q}{p}$ is the filling factor

S_e, S_h are the topological shifts

$$\boxed{N_{orb} = \frac{1}{\nu} N_e - S}$$

rational number

$S = S_h - S_e \cdot \frac{p}{q}$

$[p, q, S_e, S_h]$ → topological indices

- For $N_{orb} < N_{orb}^{*}$, there are only gapped excitations (incompressibility)
- For $N_{orb} = N_{orb}^{*}$, a unique ground state ($g=0$) (with topological degeneracy D^{g})
- For $N_{orb} > N_{orb}^{*}$, gapless excitations (quasihole excitations ~ edge excitation)

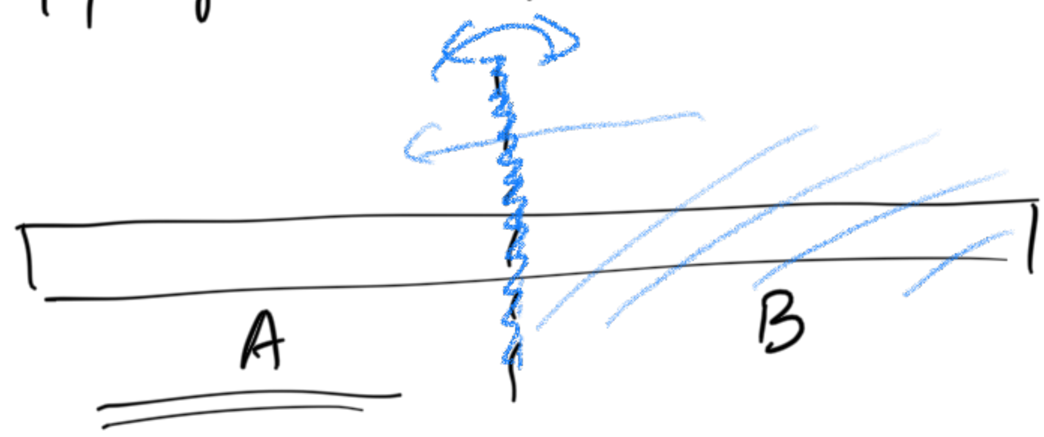
$\{p, q\}$ → filling factor → Hall conductivity

$\{S_e, S_h\}$ → Hall viscosity, (dipole moment at

$\{x_i, y_i\}$ → Hall viscosity (dipole moment ... the edge)

- Other topological indices

  · Topological Entanglement entropy



Reduced density matrix from the ground.

$$\rho_A = \sum_i \lambda_i |\Psi_{iA}\rangle\langle\Psi_{iA}|$$

$$S_A = -\sum_i \lambda_i \log \lambda_i = \alpha L - \gamma + O(L^{-1})$$

L: the lengths of boundary

$\lim_{L\to\infty} O(L^{-1}) = 0$

$\gamma = \log D$

total quantum dimension, $D = \sqrt{q}$ for Laughlin states at $\nu = \frac{1}{q}$

related to the ground state degeneracy on g-genus surface

· Central charge (Thermal Hall coefficient)

  - Edge density of state (Heat capacity of the edge)

  - Counting of the gapless edge/quasihole excitations

  - Ground state degeneracy on the torus

- "Elementary excitations" (quasiholes)
  - Fractional charge and statistics
  - Fractional spin
  - Non-abelian statistics

$\Rightarrow$ states are not unique (with degeneracy d)

$\underline{\underline{e^{i\hat{\phi}}}} \rightarrow$ $\hat{\phi}$ is a matrix of $d \times d$ dimension

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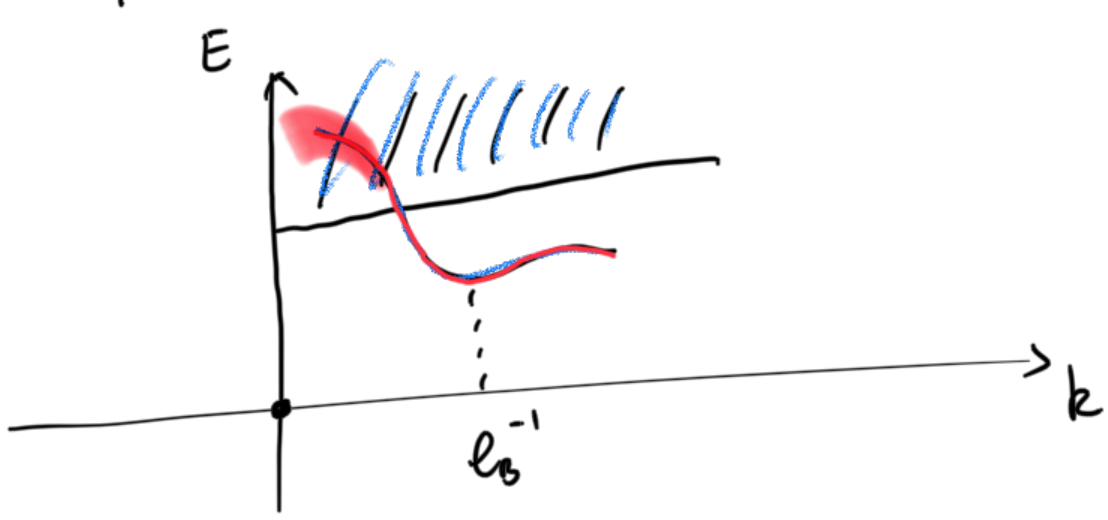
- Techniques for understanding FQHE



- No kinetic energy
- Long wavelength $\neq$ low-energy
- Perturbation and renormalization techniques no longer apply

("Rigorous") – Model Hamiltonians and model wavefunction microscopic theory

can be solved partially

- Jack polynomial and LEC formalism
  - Algebraic approach (structure of the Hilbert space)

(Good for non-abelian states, not so good for hierarchy state)

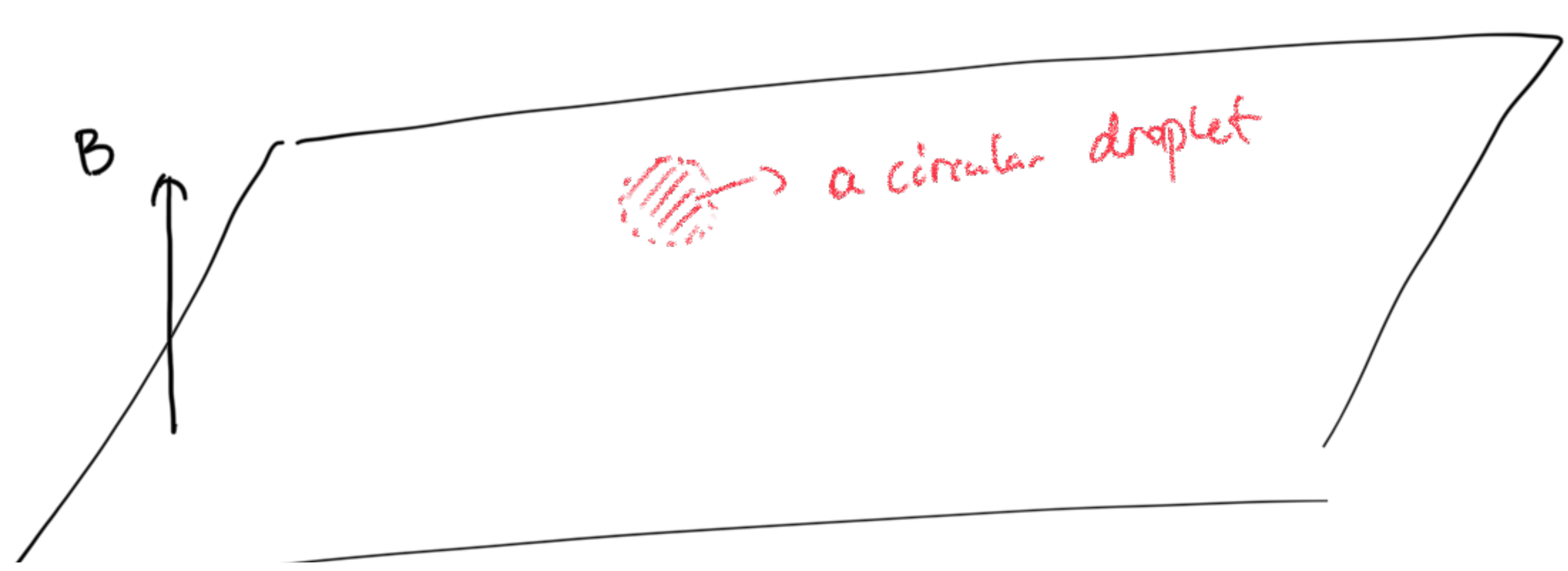
phenomenological microscopic theory - The composite fermion theory
  - FQHE → IQHE
  - Parton theory

(Good for hierarchical state, not good for non-abelian states)

Effective theory
- The CFT construction
  - Bulk-edge correspondence
- The topological field theory
  - Chern-Simons theory

- The local exclusion conditions (LEC)   PRB 100, 241302
  PRL, 125, 176402

(A simple, intuitive way of understanding FQHE both conceptually and numerically)



A two-dimensional quantum fluid

- I would like to measure the number of electrons in a circular droplet.
  - Each magnetic flux occupies $a_f = 2\pi l_B^2$, $l_B = \sqrt{\frac{\hbar}{eB}}$
  - A circular droplet of area $A = 2\pi \cdot n\, l_B^2$ contains n magnetic flux, so it can contain at most n electrons, and at most n holes

- A trivial constraint

$$\hat{C} = [\underbrace{n}_{\text{no. of fluxes}}, \underbrace{n}_{\text{max no. of electron}}, \underbrace{n}_{\text{max. no. of holes}}] \rightarrow \text{integer quantum Hall effect.}$$

- A general constraint

$$\hat{C} = [n, n_e, n_h], \quad n_e \le n, \quad n_h \le n$$

- The Laughlin state at $\nu = \frac{1}{2m+1}$

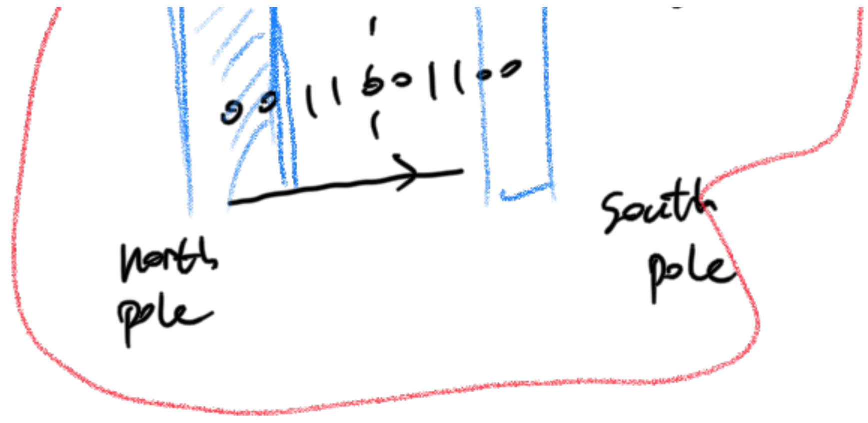
$$\Psi \sim \prod_{i<j} (z_i - z_j)^{2m+1} \sim J_\lambda^\alpha = \sum_{\mu \le \lambda} C_{\mu} m_\mu$$

$$\alpha = -\frac{2}{2m-1}$$

$$\lambda = 1\underbrace{00\ldots0}_{2m}1\underbrace{00\ldots0}_{2m}1 \cdots 1\underbrace{00\ldots}_{2m}1$$

Example:

$$1001001001 \rightarrow \lambda$$
$$0110001001 \rightarrow \mu_1$$
$$0110000110 \qquad \mu_2$$



$\hat{c} = [n, n_e, n_h]$

for $n=1$, $n_e = n_h = 1$ (trivial)

for $n=2$, $n_e = 1$, $n_h = 2$ (non-trivial)

In general for $\nu = \frac{1}{2m+1}$, the ground state satisfies $\hat{c} = [2m, 1, 2m]$

The ground state is rotationally invariant, so the constraint is satisfied by any circular droplet of $A = 2\pi n \ell_B^2$ in the quantum fluid (different from Jack polynomial)

100100100100

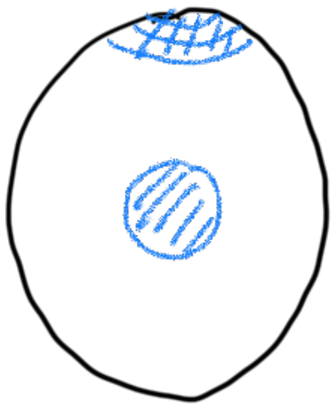
↳ for every 3 consecutive orbitals no more than 1 electrons

- The Moore-Read state

$$\Psi \sim \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$

$$= J_\lambda^\alpha \qquad \alpha = -3/2$$

$\lambda = 1100\ 1100\ 1100 \cdots 110011$



The smallest non-trivial LEC is $\hat{c} = [3, 2, 3]$

satisfied by the ground state

- What happens if we only impose the following two conditions to a quantum fluid:

  a. Satisfies a certain $\hat{c} = [n, n_e, n_h]$

  b. carrying good quantum number (momentum / angular momentum)

- For $\hat{c} = [2m, 1, 2m]$, only quantum fluids satisfying a and b are the Laughlin ground state and the quasihole states (null space of V_i pseudopotentials)

  - For $N_{orb} < (2m+1) N_e - 2m$
    no valid states

  - For $N_{orb} = (2m+1) N_e - 2m$
    a unique state (Laughlin model state)

  - For $N_{orb} > (2m+1) N_e - 2m$
    all valid states are Laughlin quasiholes

$[p, q, S_e, S_h] = [1, 2m+1, 0, 2m]$

$\hat{c}^* = [2m, 2m, 1] \to$ anti-Laughlin

- For $\hat{C} = [3, 2, 3] \to$ Moore-Read ground state and quasihole states (Pfaffian

$$[p, q, S_e, S_h] = [2, 4, 0, 2]$$

$\hat{C}' = [3, 3, 2] \to$ anti-Pfaffian

- For $\hat{C} = [n, m, n]$, we have the following results

  - $N_{orb} < \frac{2n-m}{m} N_e - 2(n-m)$

    no valid state

  - $N_{orb} = \frac{2n-m}{m} N_e - 2(n-m)$

    a unique state

  - $N_{orb} > \frac{2n-m}{m} N_e - 2(n-m)$

    a collection of quasihole states with unique counting

$$\Rightarrow [p, q, S_e, S_h] = [m, 2n-m, 0, 2(n-m)]$$

- $\hat{C}$ or a combination of $\hat{C}$'s is the ONLY physical input needed for determining the topological phase.

  - The topological indices $[p, q, S_e, S_h]$

  - Counting of the quasiholes

- Laughlin states
- Read-Rezayi Series
- Other model Ham / CFT states
  (Gaffnian, Haffnian)

* Jain states around $\frac{1}{2n}$ (especially for $n>1$)

* Interacting CF states

- Many other FQH states not discussed in the Literature

PRB 100, 245303

$\hat{C}_F = [4,3,4]$ 111001100110011100011

$\hat{C}_F^{*} = [4,4,3] \to$ anti-Fibonacci

000110001100011000

Thomale ~~state~~

$\hat{C} = \hat{C}_1 \wedge \hat{C}_2$

- Simple numerical scheme for computing FQH wavefunctions from LEC

  - Example: The Haffnian state at $\nu = \frac{2}{6}$

    $[p,q,S_e,S_h] = [2,6,0,4]$

    (For Laughlin State at $\nu = \frac{1}{3}$

    $[p,q,S_e,S_h] = [1,3,0,2]$)

  - The zero energy states of

    $$\hat{H} = \hat{V}_3^{3body} + \hat{V}_5^{3body} + \hat{V}_6^{3body}$$ ✓✓

  - Not a Jack polynomial.

· With LEC $\underline{c = [4, 2, 4]}$

For $\underline{N_e = 10}$ on the sphere, $N_\phi = 26$

The ground state is in the $L_z = 0$ sector

a. $\; 111111\,000 \cdots 00111111$ $\qquad \underline{\underline{L_z = 0}}$

The full Hilbert space

b. Remove all basis with more than 2 electrons in the left most 4 orbitals

$1100 \cdots \cdots 0011$

Truncated Hilbert space

c. Diagonalise $\underline{\underline{L^2}} \sim L^+L^- + L^-L^+$ operator within the truncated Hilbert space

$\Rightarrow$ A unique $\underline{\underline{L = 0}}$ state : the Haffnian model ground state

(We only look at states $|\psi\rangle$ with $\underline{L^+|\psi\rangle = 0}$)

On the disk,

$L^+ \sim \sum_i b_i^+$

$\boxed{\sum_i b_i^+ |\psi\rangle = 0}$

Highest weight condition