Nested Tensor Network Method and its applications

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2018. 07. 11

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Brief history about tensor network state

Strongly correlated systems is difficult:

- Analytically: non-perturbative, no obvious small parameters
- Numerically: Exponential wall: degree of freedom grows exponentially with system size

Weak Coupling Approach: Suitable for weak-coupling systems

- Convert a many-body problem into single-body: mean field theory, density functional theory

Strong Coupling Approach:

- keeps only a finite set of many-body basis:
- Configuration interactions(CI), Coupled Cluster Expansion, QMC, Numerical RG.

✓ AKLT authors(1987): prototype of matrix product state and honeycomb tensor-network

✓ **M. Fannes(1991):** MPS in name of Finitely Correlated State (FCS), and Tree Tensor Network state (TTN)

✓ Niggemann: special TNS for honeycomb Heisenberg model, equivalence between exp. cal. and classical PF

✓ Ostlund and Rommer (1995): DMRG (1992)'s wavefunction is a MPS, area law

✓ Sierra and Martin-Delgado: general wavefunction ansatz to study a quantum lattice model

✓ Nishino: in name of Tensor Product State (TPS), general variational ansatz to study 3D classical model

Why do we need tensor renormalization?

- Projected Entangled Pair State (PEPS, 2004)
- Multi-scale Entanglement Renormalization Ansatz (MERA, 2007)
- Correlator Product State (CPS, 2009)
- Projected Entangled Pair State (PEPS, 2014)

✓ Density Matrix Renormalization Group

- Best method for 1D quantum model
- Violet area law
- 2D->1D, artificial long-range interaction
- Hope: extrapolation, even gapped case

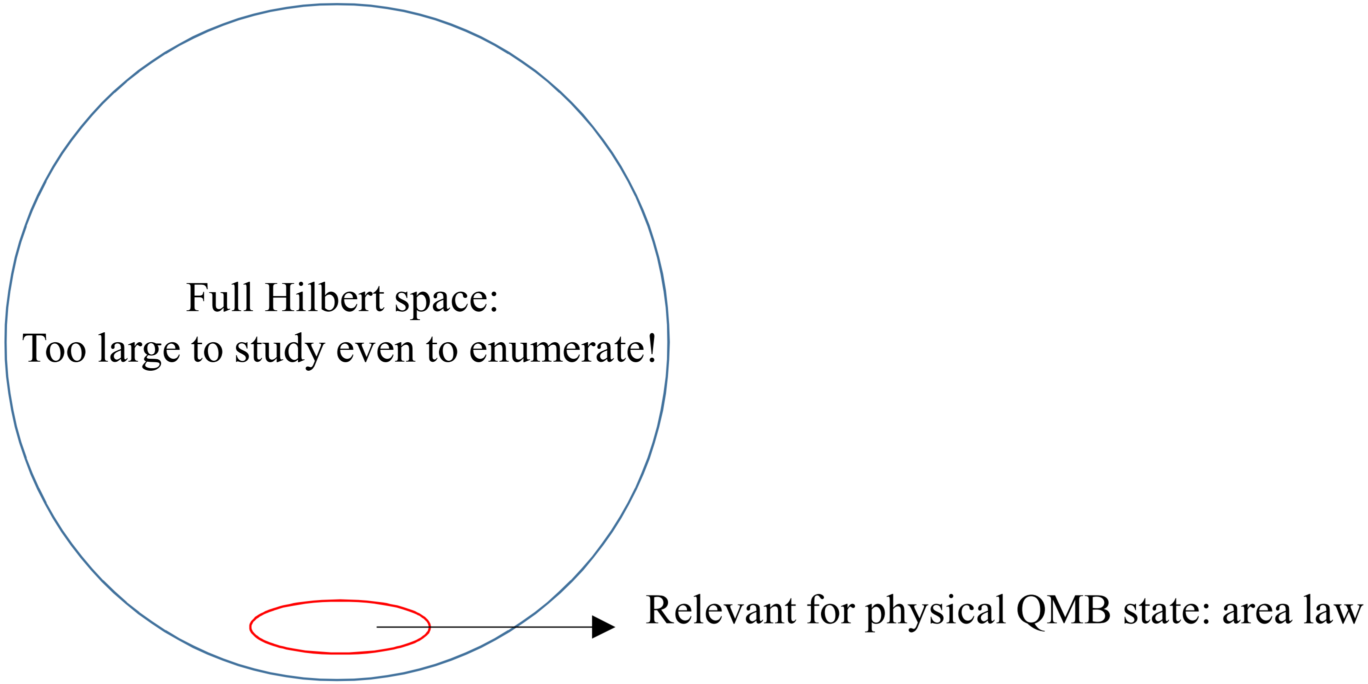
✓ Quantum Monte Carlo

- No dimension consideration
- Suffer from the "minus-sign" problem for fermion and frustrated spin system

A possible direction: Tensor Renormalization (TNS, TNM)

Why renormalization is possible?

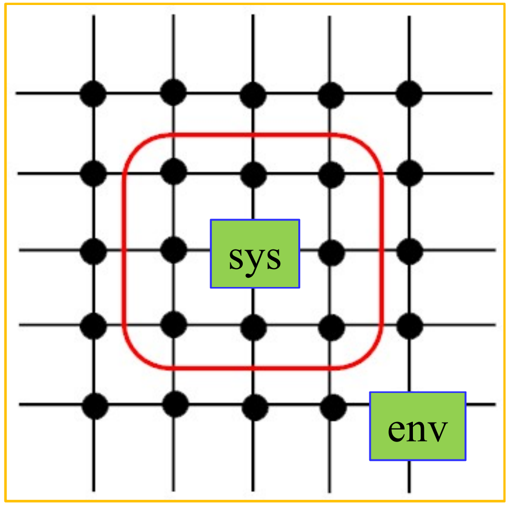
- Hilbert space is compressible:



Ref: D. Poulin, A. Qarry, R. Somma, F. Verstraete, Phys. Rev. Lett. 106, 170501 (2011)

About the corner: Area law of Entanglement Entropy

- (Boundary) Area Law in quantum information: for a gapped system with local H



$$S \sim L^{d-1} \sim \log D_{min}$$

D_{min} : N_{min} of basis needed to describe the grdst (entanglement entropy) faithfully.

$d = 1$: $D_{min} \sim const$
$d = 2$: $D_{min} \sim e^L$

•1D: local gapped Hamiltonian with only constant degeneracy of ground state
•Quasi-free (i.e., quadratic int) boson and fermion gapped Hamiltonian: in any D
•Known violation: 1D critical fermion has log correction, 2D critical fermion suggests log correction
1D critical XY chain (i.e., h<=2 in isotropic case, h=2 in anisotropic case)
•**General belief**: ground state of local gapped Hamiltonian obeys.

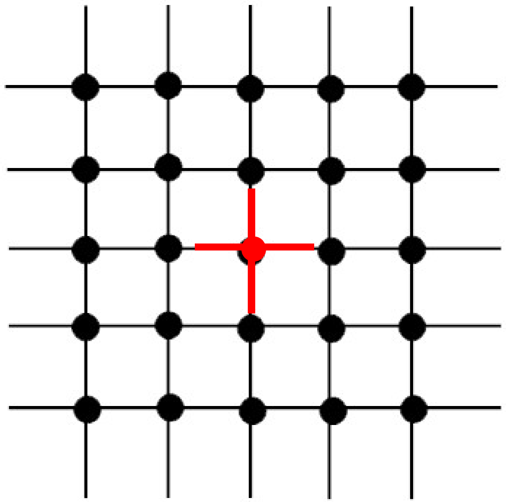
Ref: J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010).

Classical system: Tensor Network Model

- all statistical models with only local interactions can be represented as *tensor-network models* and effectively evaluated, defined on *real lattice or dual lattice*.

$$Z = Tr\prod_i T_{x_i y_i z_i w_i}$$

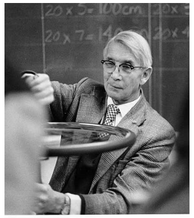
Tensor-network model in real lattice



$$H = -J\sum_{\langle ij\rangle} S_i S_j$$

$$Z = \sum_{S_1 \dots S_N} \exp\left(\beta \sum_i S_i S_{i+1}\right)$$
$$= Tr\left(A \cdots A\right)$$
$$= \lambda_{\max}^N \qquad N \to \infty$$

$$A = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$



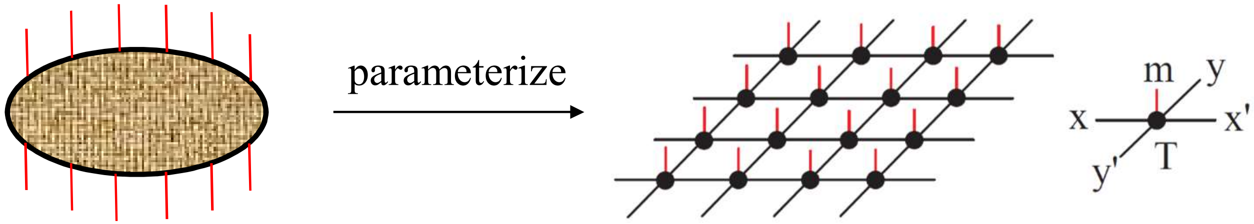
Ernst Ising

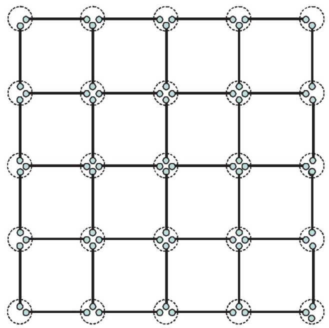
Ref: H. H. Zhao, Z. Y. Xie, Q. N. Chen, Z. C. Wei, J. W. Cai, and T. Xiang, Phys. Rev. B 81, 174411 (2010)

Quantum lattice system: Tensor Network State (TNS)

- *Tensor network state* provides a faithful representation of the ground state wavefunction of a quantum lattice model that satisfies the area law of the entanglement entropy.
- A kind of construction, e.g., PEPS: $d^N \to N D^4 d$

Projected Entangled Pair State





$$|\Psi\rangle = \sum_{\{m\}} C_{m_1 m_2 \ldots} |m_1 m_2 \ldots\rangle$$

$$|\Psi\rangle = Tr \prod T_{x_i x_i' y_i y_i'}[m_i] |m_i\rangle$$

Virtual auxiliary state
Bond dimension (D)

Real physical state

- 1D case: Matrix Product State (MPS), or Tensor Train, DMRG wavefunction

✓ Area law: is believed to be a faithful representation of grdst of ***local gapped*** H.

✓ Formally has no sign problem, can encode fermion sign

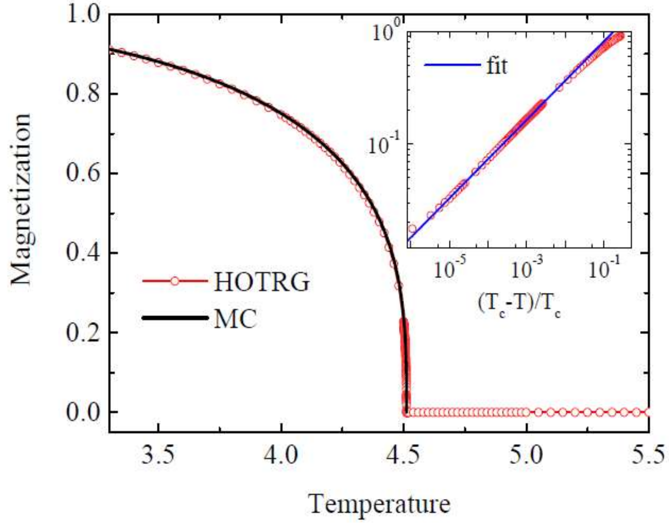
✓ Can show power-law correlation function, at finite D

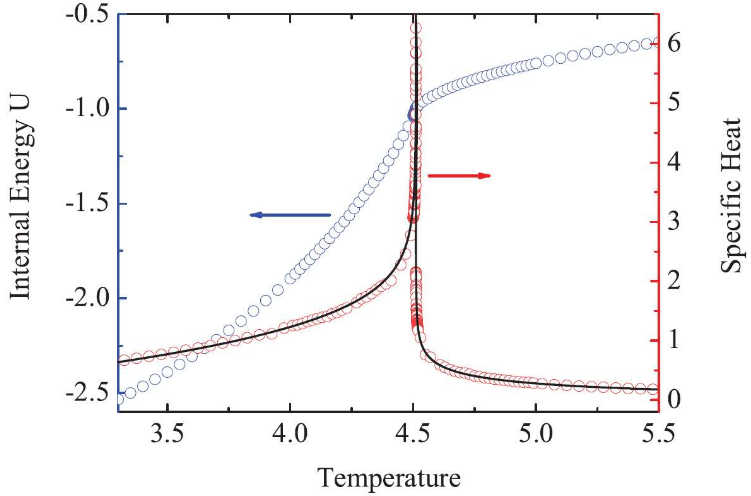
Successful examples: Ising model on cubic lattice

ZYXie, PRB 86, 045139 (2012)

SWang, et al, CPL 31, 070503 (2014)

- Critical point accuracy:





alpha(+): 0.1023

alpha(-): 0.1137

gamma: 0.3295

method	T_c
Monte Carlo [33]	4.511523
Monte Carlo [25]	4.511528
Monte Carlo [35]	4.511525
Monte Carlo [36]	4.511516
Series Expansion [34]	4.511536
KWA [41]	4.5788
CTMRG [13]	4.5704
CTMRG [40]	4.5392
TPVA [14]	4.554
Algebraic variation [37]	4.547
HOTRG(D=16, from U)	4.511544
HOTRG(D=16, from M)	4.511546

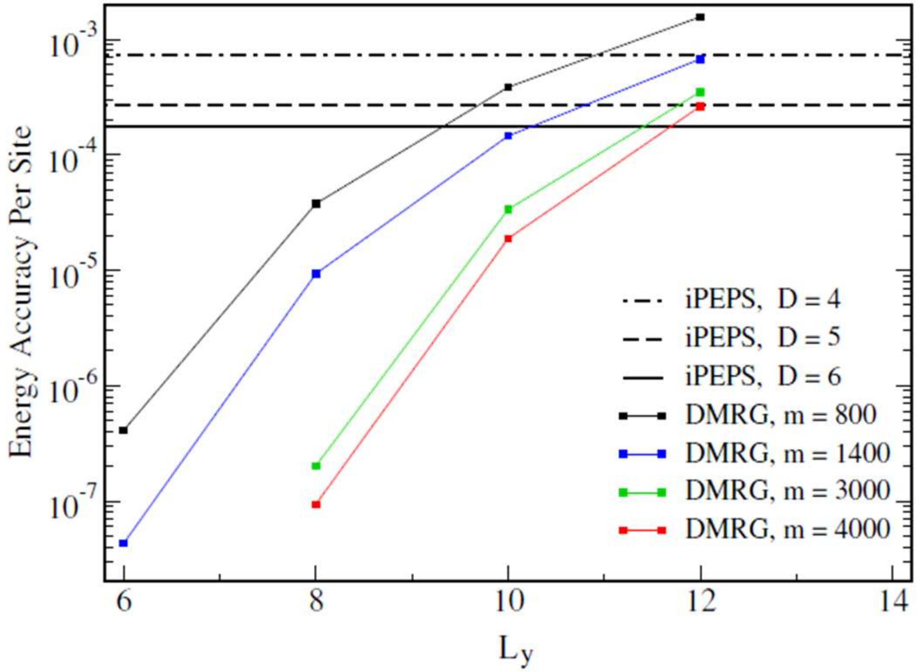
former NRG

Monte Carlo (2003)[43]	4.5115248(6)
Monte Carlo (2010)[44]	4.5115232(17)
HOTRG ($D = 16$) (2012)[24]	4.511544
HOTRG ($D = 23$, this work)	4.51152469(1)

Successful examples: Heisenberg model on square lattice

S.R.White, et al, Annu. Rev. CMP 3, 111(2012)

- spin-1/2 AF Heisenberg model on square lattice: PEPS



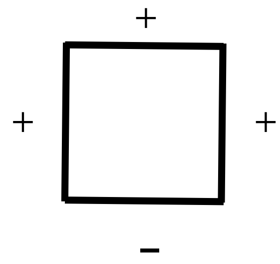
Energy accuracy
VS
Cylinder width

DMRG:
jumps fast as width(Ly) grows
PEPS:
improves as D grows

6 > 4000

Note:
iPEPS reference: VMC extrapolation in Sandvik PRB 56, 11678(1997)
DMRG reference: DMRG extrapolation in truncation error at given size.

Edward-Anderson model on square (with PBC)



$$H(\underline{\sigma}) = -\sum_{(ij)\in E} J_{ij}\sigma_i\sigma_j - \sum_{i\in V} h_i\sigma_i$$

$$P(J) = p\delta_1 + (1-p)\delta_{-1}$$

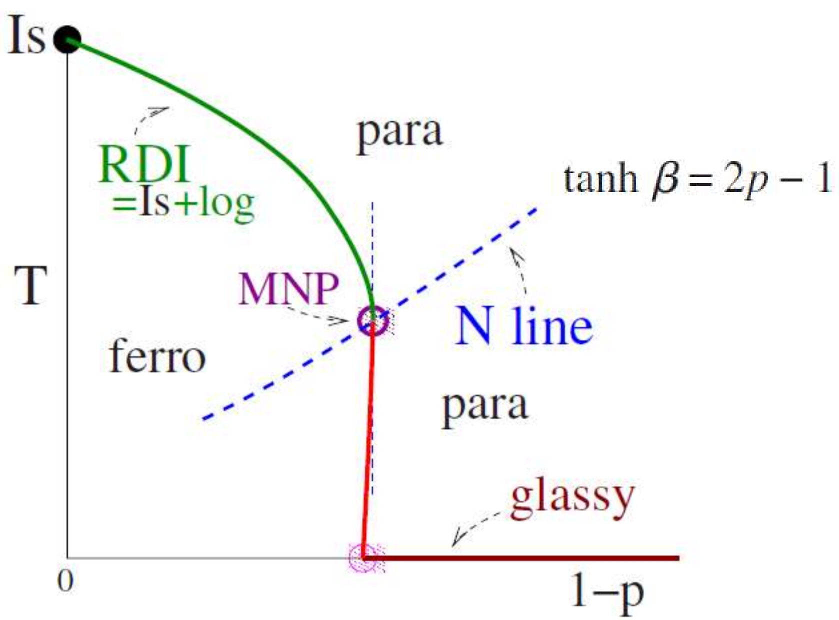


TABLE I. Location of the MNP.

Methods	p^*
BP [28]	0.79
GBP [27,28]	0.85
Duality analysis [32]	0.889 972
Duality analysis [33]	0.890 813
pTRG	0.890 830(22)
Monte Carlo [29]	0.890 81(7)
Monte Carlo [43]	0.890 83(3)

PRB 90, 174201 (2014)

Deep Learning: ResNet

Our group, *to appear soon*

- Additional parameters:

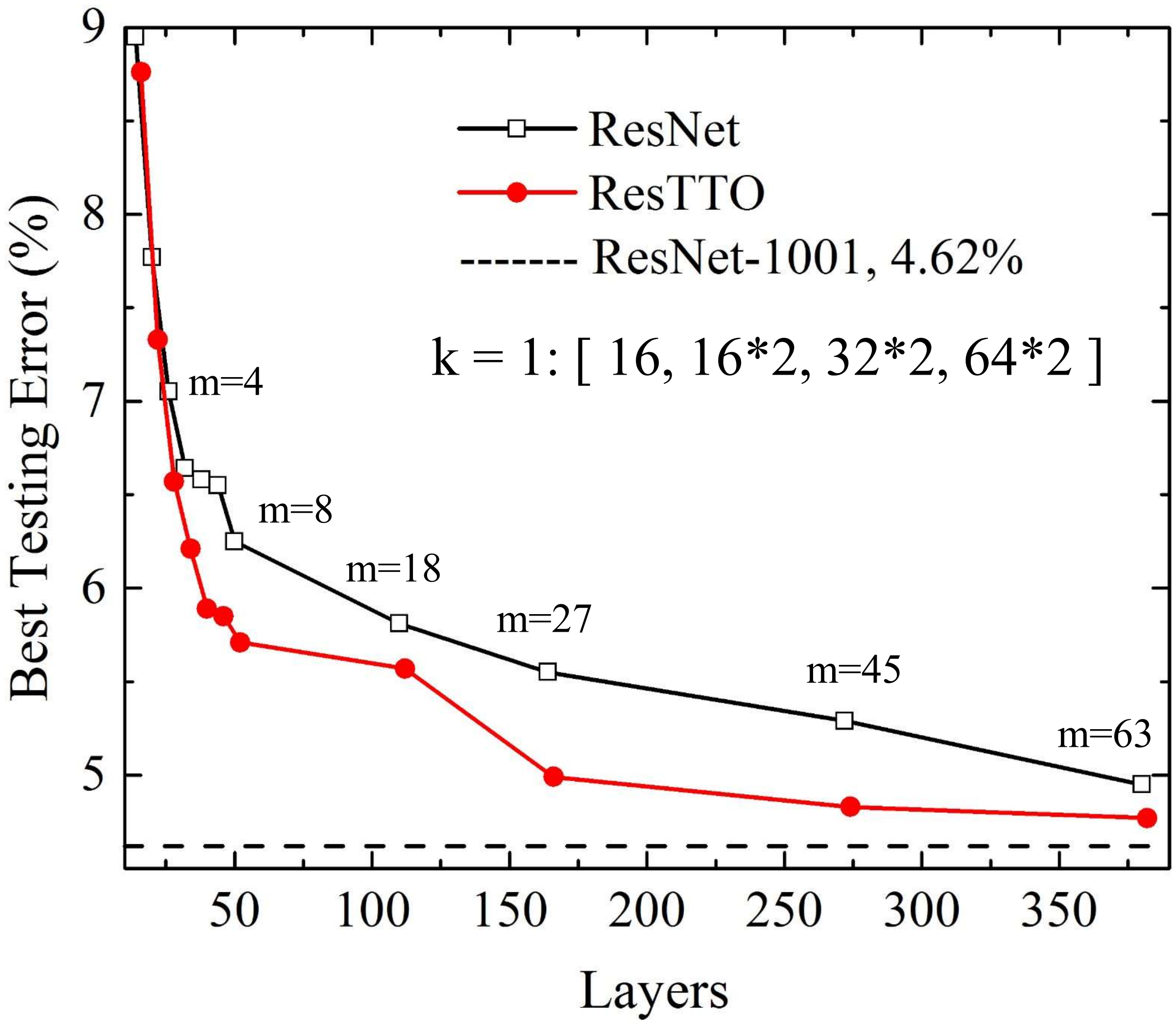
  k=1, Na ~ 2300

  k>1, Na ~ 2700 + $8k^2$

- ResNet parameters:

  N ~ 400 + 2950*k + (9.6*m-2.5)*10^4 *k^2

m	Layers	Na / N	Best testing error
4	28	0.63%	7.05% -> 6.57%
8	52	0.31%	6.25% -> 5.71%
18	112	0.13%	5.81% -> 5.57%
27	166	0.09%	5.55% -> 4.99%
45	274	0.05%	5.25% -> 4.83%
63	382	0.04%	4.95% -> 4.77%

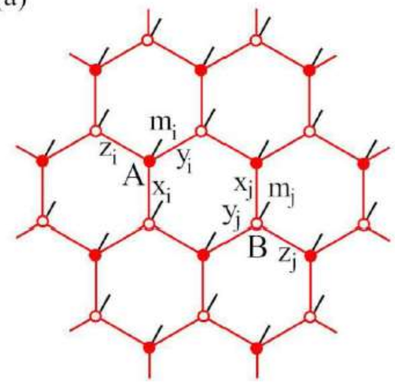


Bottleneck

- Quantum lattice model:

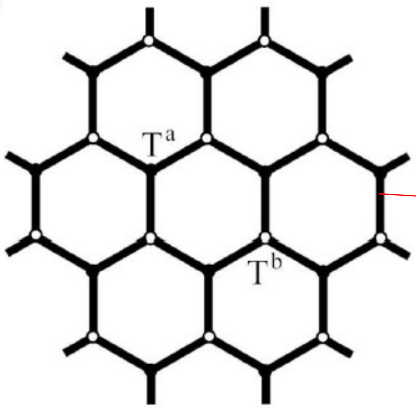
1. determine the tensor-network representation of targeted wave-function: time-evolution, energy minimization

(a)



$$|\Psi\rangle = \mathrm{Tr} \prod_{i\in A, j\in B} A_{x_i y_i z_i}[m_i] B_{x_j y_j z_j}[m_j] |m_i m_j\rangle$$

(c)



Bond dimension: D^2

- Time evolution: $e^{-\beta H}|\Psi\rangle = \sum_i e^{-\beta E_i}|\phi_i\rangle \xrightarrow{\beta\to\inf} e^{-\beta E_0}|\phi_0\rangle$

how to update/renormalize after a small evolution step

$$||\Psi_f\rangle - e^{-\tau H}|\Psi_i\rangle|$$

(1). Simple update (entanglement mean field approximation)

✓ Use the local entanglement spectra as effective environment

✓ local tree approximation which can be solved by SVD/HOSVD

(2). Full update (Global variation)

Solve the linear equation iteratively $NX = M^e$

(3). Cluster update:

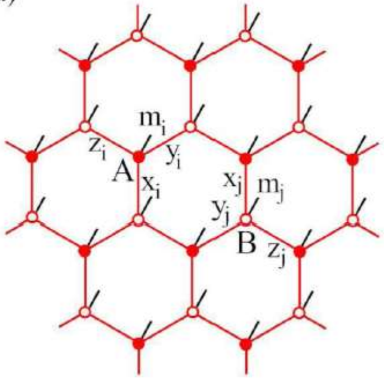
Use small cluster and its mean field as effective environment

Bottleneck

- Quantum lattice model:

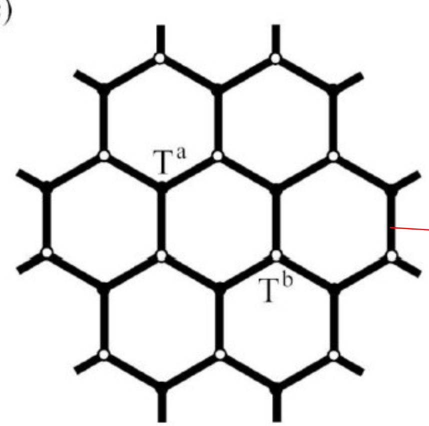
1. determine the tensor-network representation of targeted wave-function:

(a)



$$|\Psi\rangle = \mathrm{Tr} \prod_{i \in A, j \in B} A_{x_i y_i z_i}[m_i] B_{x_j y_j z_j}[m_j] |m_i m_j\rangle$$

(c)



Bond dimension: D^2

- Energy minimization (global extremum problem):

find a PEPS $|\Psi\rangle$ which minimize the energy:

$$\frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

This can be done equivalently as an optimization problem

$$\min_{|\Psi\rangle \in \text{family}} \left(\langle \Psi | H | \Psi \rangle - \lambda \langle \Psi | \Psi \rangle \right)$$

which can be reduced to generalized eigenvalue problem

$$H^e X = \lambda N X$$

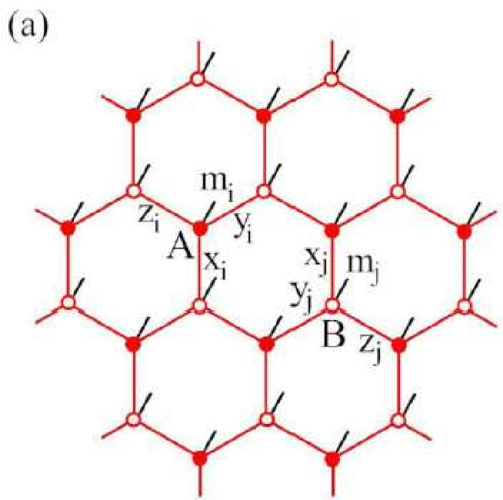
Bottleneck

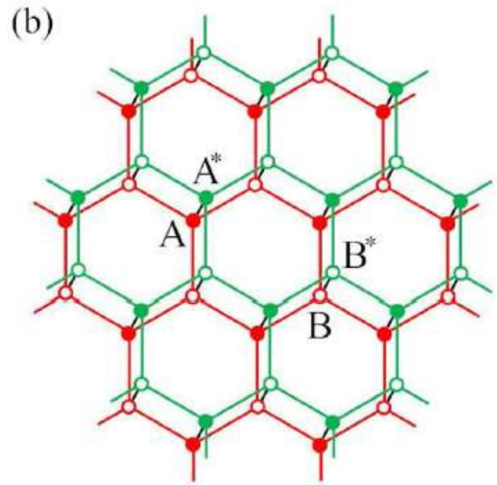
2. contract the network to get the expectation value

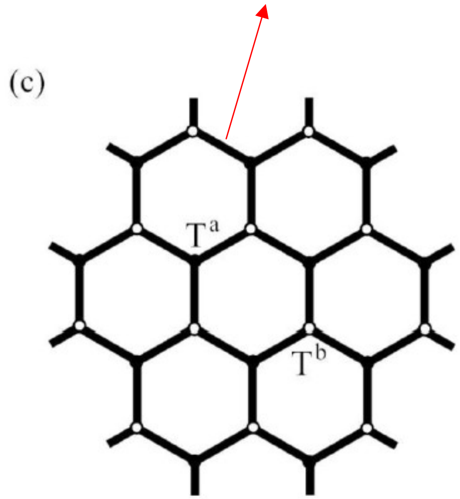
Green: $\langle\Psi|$
Red: $|\Psi\rangle$

$$\langle O\rangle = \frac{\langle\Psi|\hat{O}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

Bond dimension: D²







$$|\Psi\rangle = \mathrm{Tr} \prod_{i\in A, j\in B} A_{x_i y_i z_i}[m_i] B_{x_j y_j z_j}[m_j] |m_i m_j\rangle$$

$$T^a_{xx',yy',zz'} = \sum_m A_{xyz}[m] A^*_{x'y'z'}[m],$$

$$T^b_{xx',yy',zz'} = \sum_m B_{xyz}[m] B^*_{x'y'z'}[m].$$

Reduced **T**ensor **N**etwork

- Renormalization: Compression of DOF (real Hilbert space, or virtual space) by discarding the irrelevant
- Contraction of **RTN** with D²: seems unavoidable, the main bottleneck of all TRG methods!

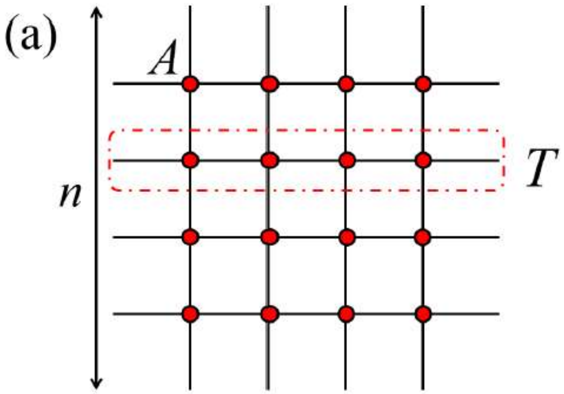
Why do we need large D and large χ

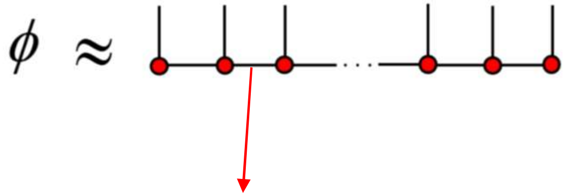
- ✓ Time Evolving Block Decimation(TEBD) / Boundary MPS

  Target: effective MPS representation of the dominant eigenvector

- ✓ Power method with truncation: not variational!

$$Z = \mathrm{Tr} T^n \simeq \lambda_{dom}^n$$





A huge and terrible task!

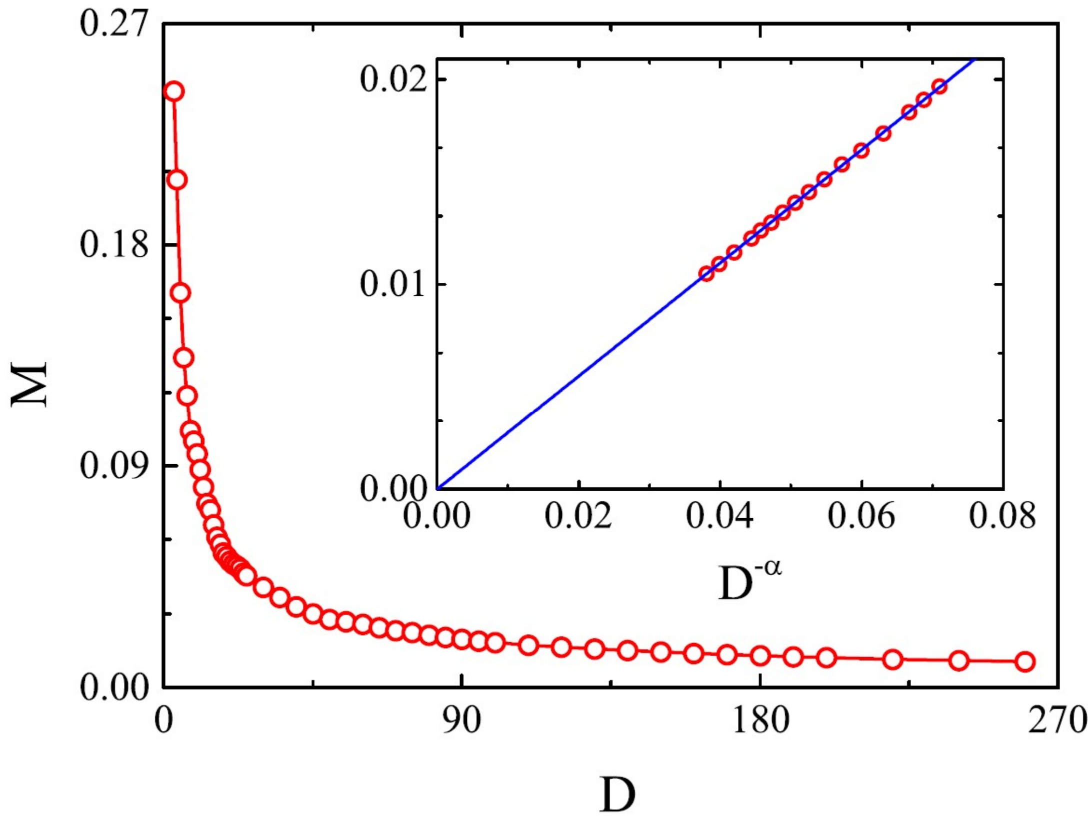
Memory: $D^4\chi^2$
CPU: $D^6\chi^3$

D=10: 750M
D=13: 6G
D=20: 190G
D=25: 1.1T

Power method to get the fixed point: $T^n|\Psi\rangle \rightarrow |\Psi_{dom}\rangle$

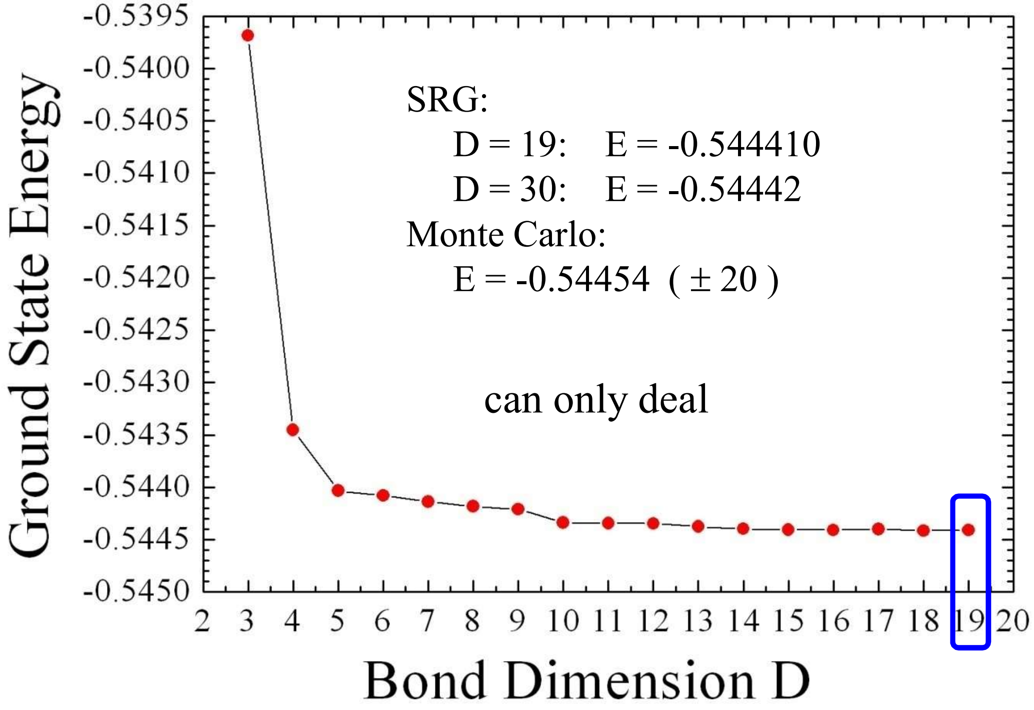
- ✓ In principle: We always need a curve of O(D), in which each point is obtained by χ-scaling ($\chi \sim D^2$).

Even though we can obtain wavefunction with very larger D



We can only deal with small D when doing expectation value calculation

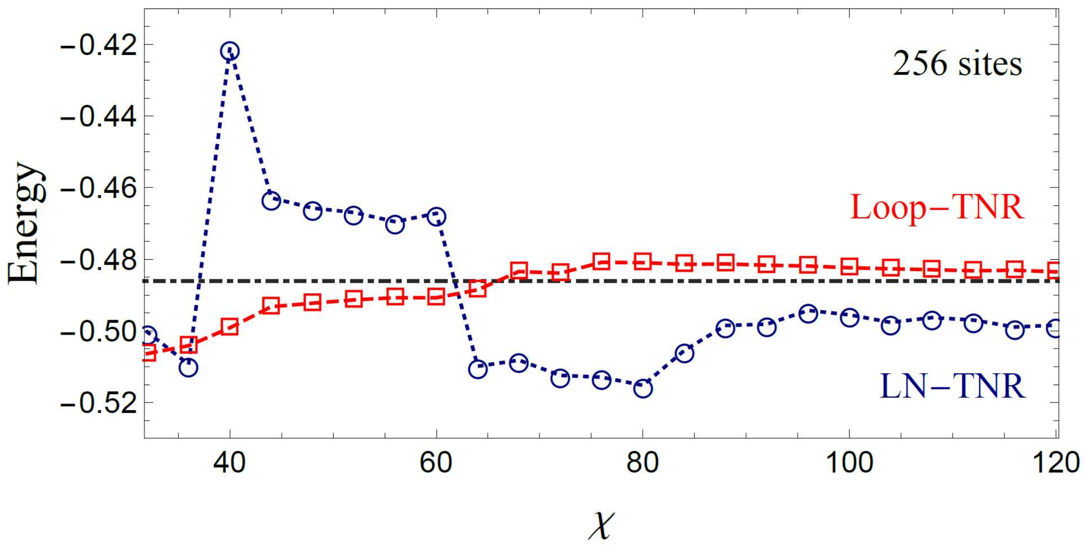
HHZhao, *et al,* PRB 81, 174411 (2010)



AF Heisenberg model on honeycomb lattice
With U(1) symmetry

$$S[x_i] + S[y_i] + S[z_i] = m_i$$

Shuo Yang, *et al,* PRL 118, 110504 (2017)



3-state RVB wavefunction
(J_1-J_2 AF Heisenberg model on square lattice): N = 256

Especially for Fermion: Larger D

Fermions with 2D tensor networks

Simulate fermions in 2D?
Before April 2009: **NO!**
Since April 2009: **YES!**

Different formulations:

P. Corboz and G. Vidal, Phys. Rev. B 80, 165129 (2009)

T. Barthel, C. Pineda, and J. Eisert, Phys. Rev. A 80, 042333 (2009)

C. V. Kraus, N. Schuch, F. Verstraete, J. I. Cirac, Phys. Rev. A 81, 052338 (2010)

Q.-Q. Shi, S.-H. Li, J.-H. Zhao, and H.-Q. Zhou, arXiv:0907.5520

P. Corboz, G. Evenbly, F. Verstraete, G. Vidal, Phys. Rev. A 81, 010303(R) (2010)

C. Pineda, T. Barthel, and J. Eisert, Phys. Rev. A 81, 050303(R) (2010)

P. Corboz, R. Orus, B.Bauer, G. Vidal, PRB 81, 165104 (2010)

I. Pizorn, F. Verstraete, Phys. Rev. B 81, 245110 (2010)

Z.-C. Gu, F. Verstraete, X.-G. Wen. arXiv:1004.2563

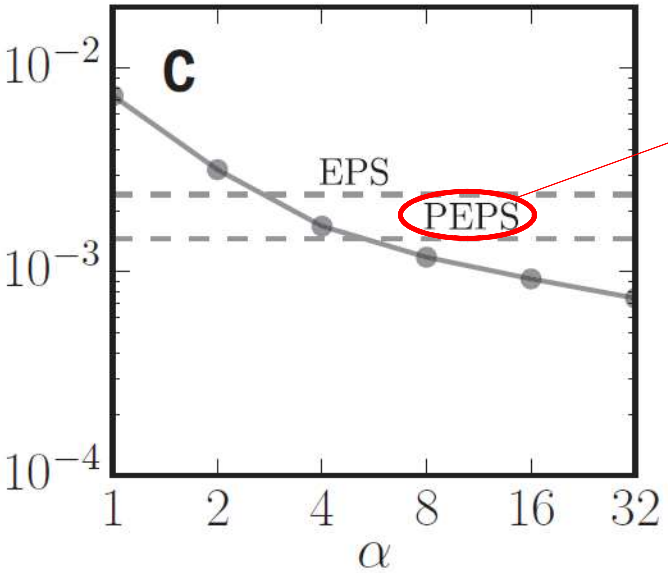
Short-range correlation in momentum space,
long-range correlation in real space,
more entanglement entropy
Fermi surface, BEC(Bose metal)

P. Corboz's contribution:
PRL 113, 046402 (2014): t-J, D*=7
PRB 93, 045116 (2016): Hubbard, D*=8
Science 358, 1155 (2017): Hubbard, D*=8~9

Even very strong misunderstanding, e.g.,

Solving the quantum many-body problem with artificial neural networks
Giuseppe Carleo and Matthias Troyer (February 9, 2017)
Science **355** (6325), 602-606. [doi: 10.1126/science.aag2302]

Editor's Summary



D<=6, in PRB 90, 064425 (2014)

dashed line). (**C**) Accuracy for the AFH model on a 10-by-10 square lattice with PBCs, compared with the precision obtained by EPS [upper dashed line (*35*)] and PEPS [lower dashed line (*36*)]. For all cases considered here, the NQS approach reaches MPS-grade accuracies in one dimension and systematically improves the best known variational states for 2D finite lattice systems.

For Monte Carlo: Larger Size is important!

Y. Iqbal, et al, arXiv: 1606.02255
T. Li, arXiv: 1601.02165

Finite size vs Thermodynamic limit
qualitatively different

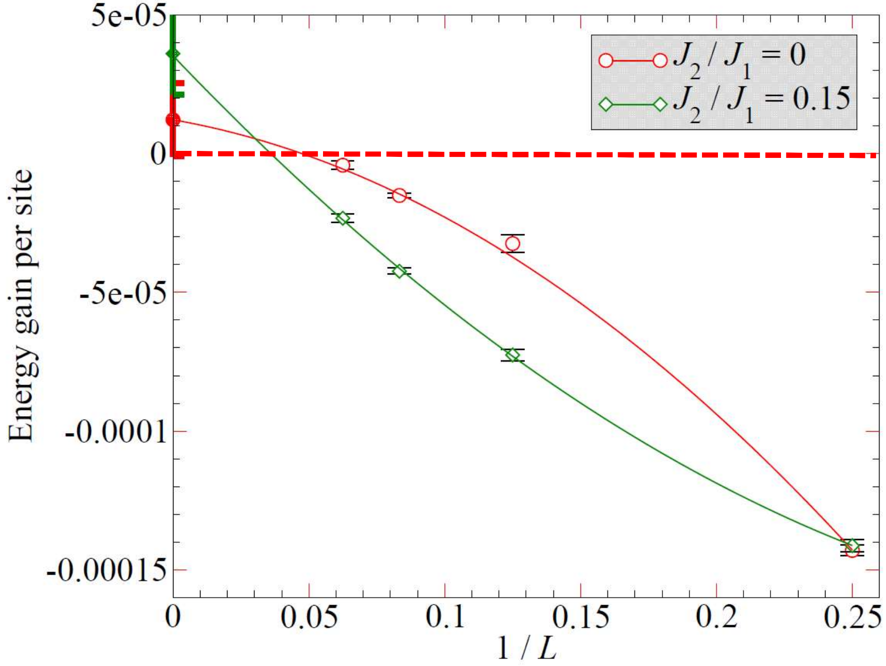
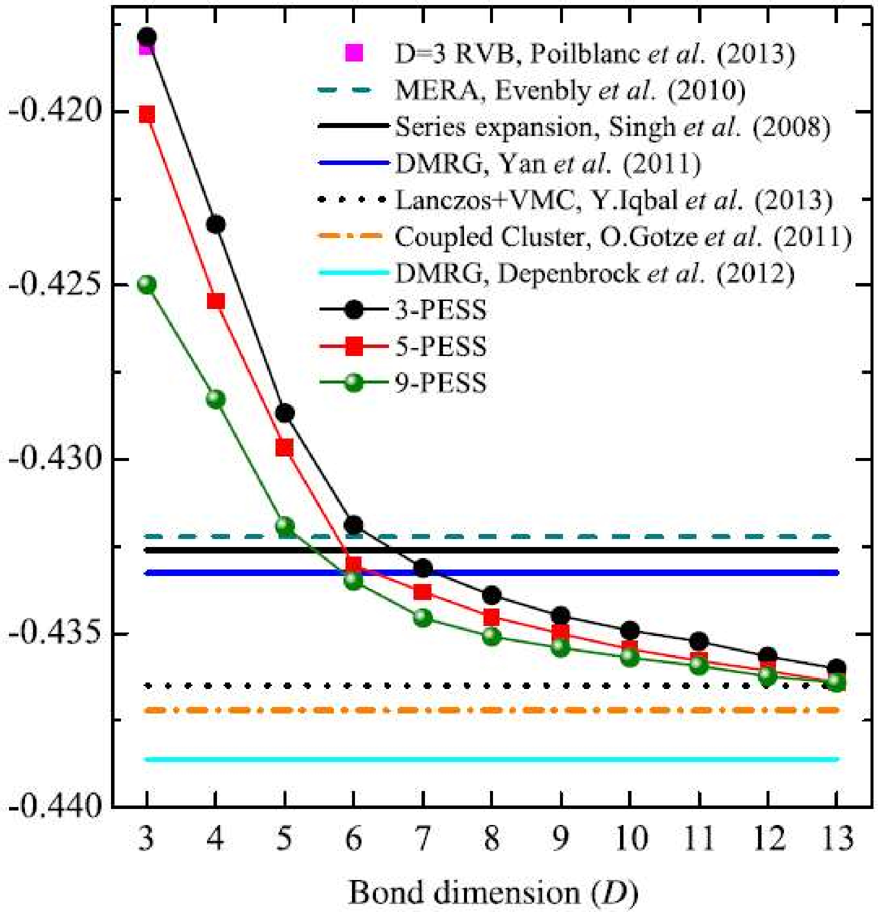
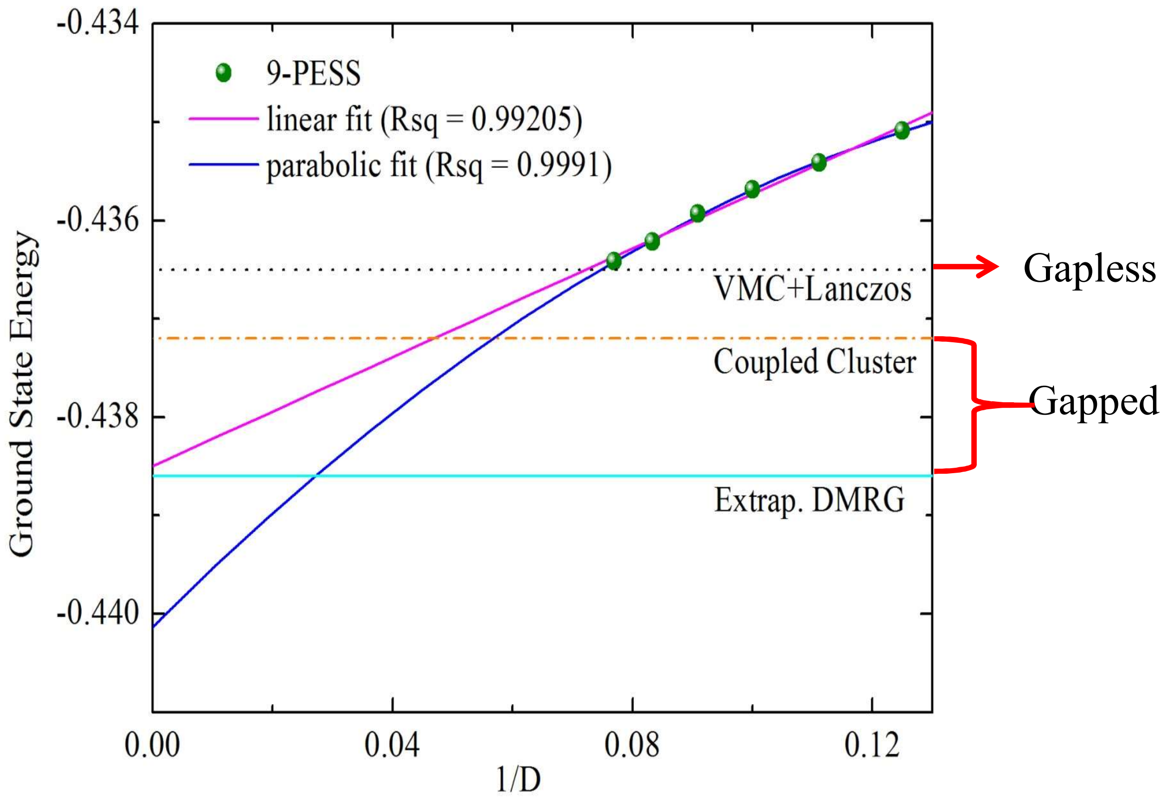


FIG. 1. The size scaling of the energy gain per site $\Delta E = E_{\mathbb{Z}_2} - E_{U(1)}$ is shown for $J_2 = 0$ and $J_2/J_1 = 0.15$, for $L = 4$, 8, 12 and 16 clusters. The results for $L = 4$ and 8 are from Ref. [4].

For TNS: Larger D is important!

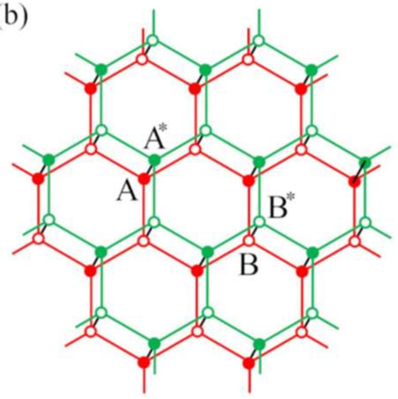




Which extrapolation is the correct one!
Gapped or gapless?

Accept ? No! Some former efforts

Green: $\langle\Psi|$
Red: $|\Psi\rangle$



- Partial summation of the physical indices: Monte Carlo sampling

[33] L. Wang, I. Pizorn, and F. Verstraete, Phys. Rev. B **83**, 134421 (2011).
[34] W.-Y. Liu, S.-J. Dong, Y.-J. Han, G.-C. Guo, and L. X. He, Phys. Rev. B **95**, 195154 (2017).
[35] H.-H. Zhao, K. Ido, S. Morita, and M. Imada, arXiv:1703.03537.
Phys. Rev. B 96, 85103 (2017)

Does not change the scaling
Accuracy is lost due to the sampling

System size is limited

- Single-layer contraction: use only **bra** or **ket**, not their product.

Iztok Pižorn, et al, Phys. Rev. A **83**, 052321 (2011)

$$\langle O\rangle = \frac{\langle\Psi|\hat{O}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

Change the target
Accuracy is greatly lost

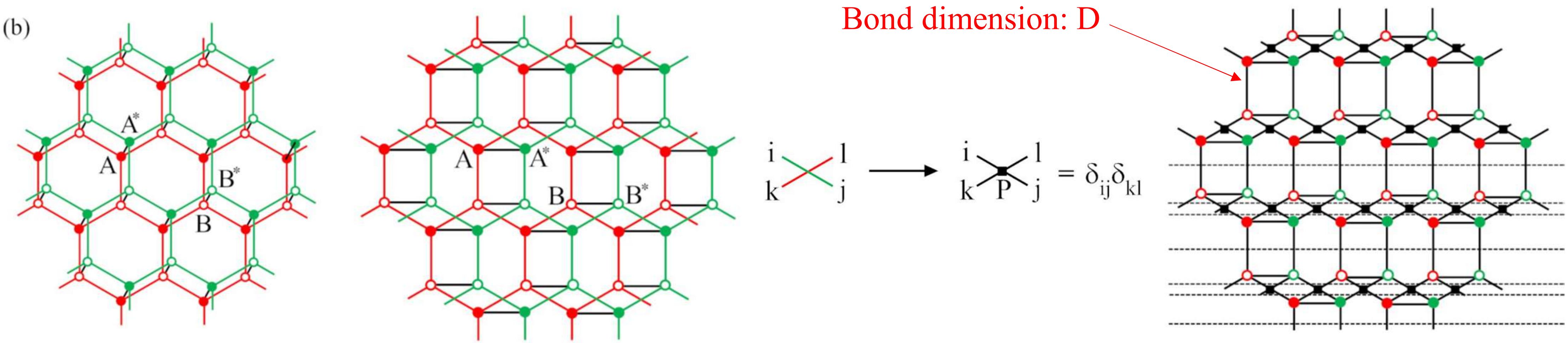
Accuracy is lost

Nested Tensor Network: Dimension Reduction Technique

Idea: physical indices are not summed over first, but remained and projected to the virtual plane

Green: $\langle\Psi|$
Red: $|\Psi\rangle$

ZYXie, *et al*, PRB 96, 045128 (2017)

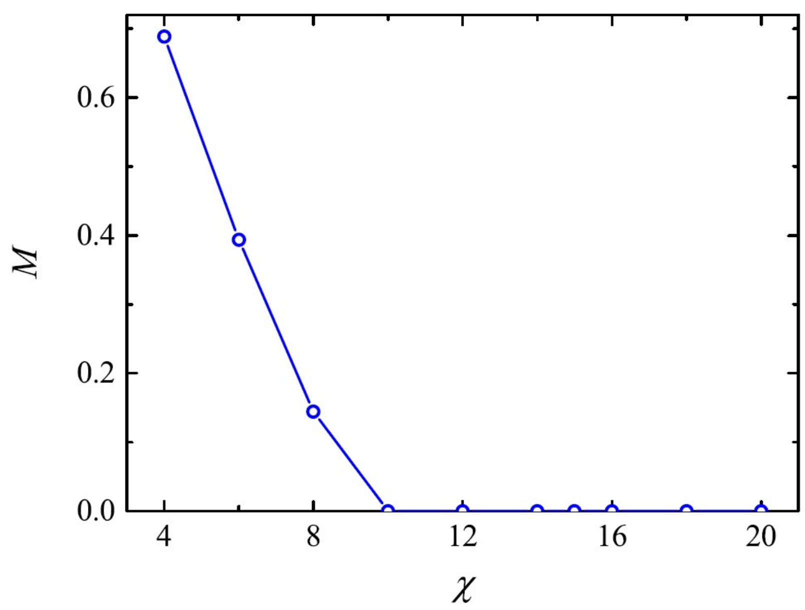


Seems trivial, but the consequence is non-trivial:

Memory: $D^4\chi^2 \rightarrow D^2\chi^2$
CPU: $D^6\chi^3 \rightarrow D^3\chi^3$
D: **10~13 → 25~30**

Does not change the target
Change the scaling essentially
Accuracy is actually improved by keeping larger χ

Validity Test: Some known state



spin-2 SSS on Kagome: D=3
E = 0, M = 0

RVB state on KAFH: D=3
E = -0.393124(1) finite-size-scaling
E = -0.393123

App1ication 1: Spin-1/2 Kagome anti-ferromagnetic Heisenberg (AFHK) model

$$H = J\sum_{\langle ij\rangle}(S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z)$$

Valence bond crystal

Singh & Huse, PRB 2008 series expansion

Evenbly & Vidal, PRL 2010 MERA

Iqbal, Becca & Poilblanc, PRB 2011 VMC

Gapped spin liquid (Topological)

Jiang, Weng & Sheng, PRL 2008 DMRG

Yan, Huse & White, Science 2011 DMRG

Depenbrock, McCulloch & Schollwock, PRL 2012 DMRG

Jiang, Wang & Balents, Nature Physics 2012 DMRG

Gong, Zhu & Sheng, Scientific Reports 2014 DMRG

Li, arXiv:1601.02165 VMC

Mei, Chen, He & Wen, arXiv:1606.09639 SU(2)-PESS

Gapless spin liquid (Algebra)

Hastings, PRB 2000

Ran, Hermele, Lee & Wen, PRL 2007 VMC

Iqbal, Becca, Sorella & Poilblanc, PRB 2013 VMC+Lanczos

Hu, Gong, Becca & Sheng, PRB 2015 VMC

Jiang, Kim, Han & Ran, arXiv:1610.02024 SU(2)-PEPS

He, Zaletel, Oshikawa & Pollmann, arXiv:1611.06238 DMRG

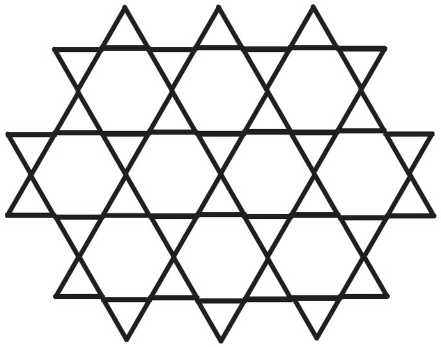
Experiment:

Nutron Scattering: tends gapless

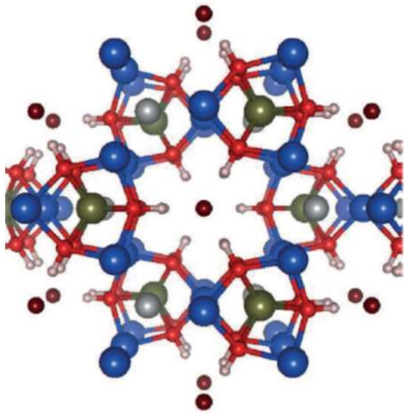
T. H. Han, *et al*, Nature(2012).

NMR: gapped ~ [0.03, 0.07]

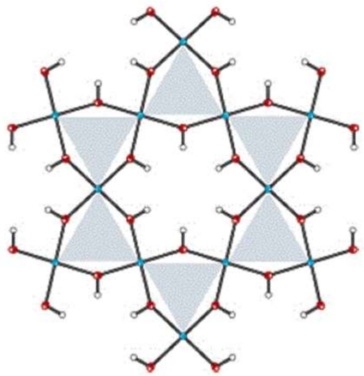
M.X.Fu, *et al*, Science(2016)



Kaomge lattice



$Cu_3Zn(OH)_6FBr$

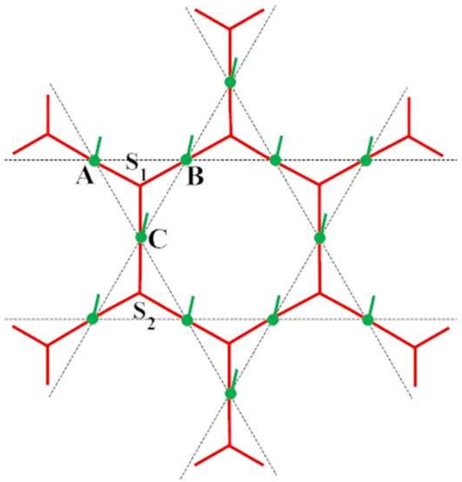


Herbertsmithite
$ZnCu_3(OH)_6Cl_2$

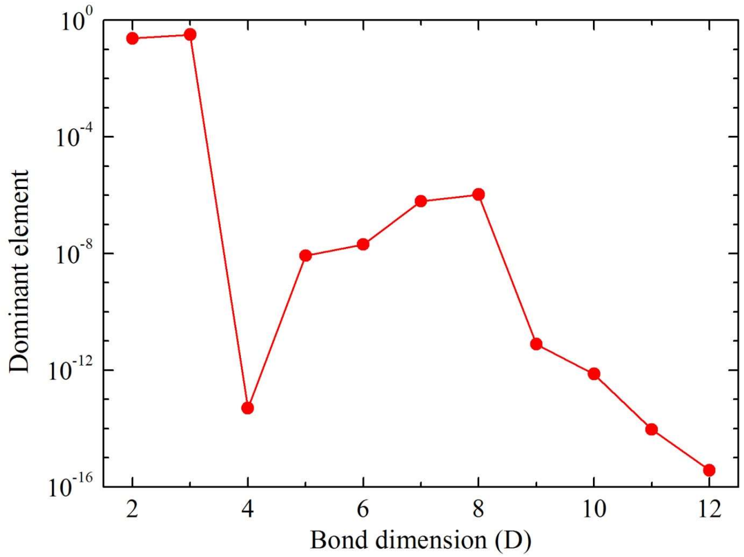
Generalization pair entanglement to simplex entanglement
Projected entangled ***simplex*** states(PESS) ansatz

ZYXie, *et al*, PRX 4, 011025 (2014).

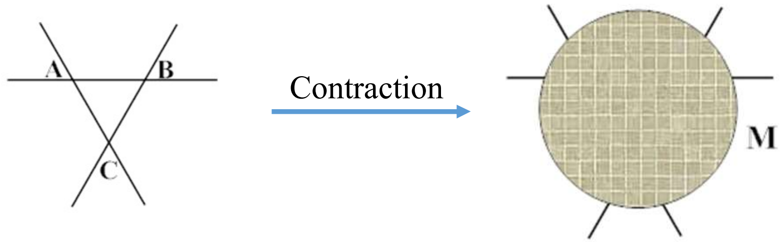
Simplex ~ possible building block, such as triangle for Kagome



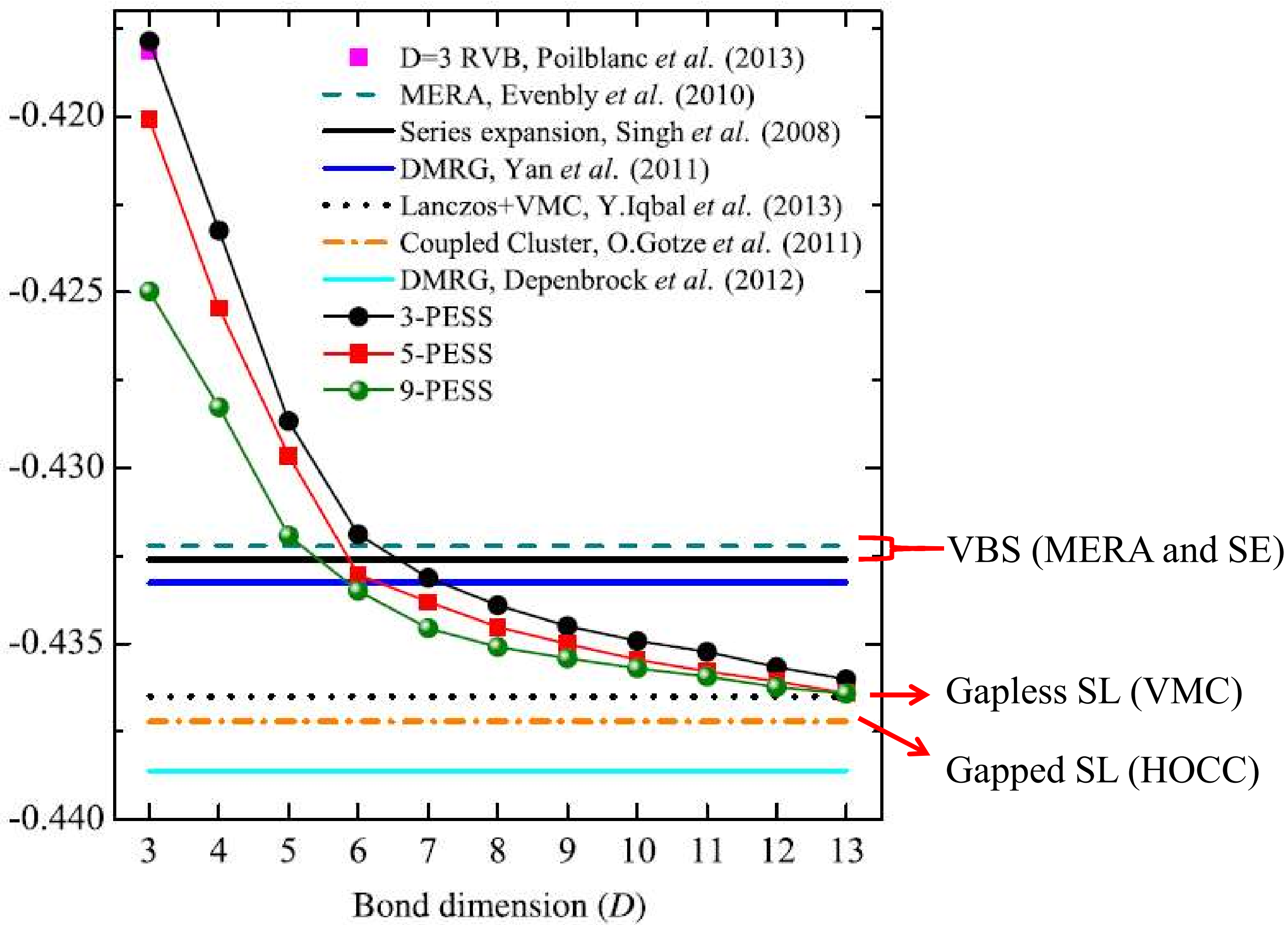
- Introduce a simplex tensor S:
  triangle/simplex entanglement, instead of pair
- defined on unfrustrated lattice:
  honeycomb, no hidden frustration here!
- A better representation for frustrated systems



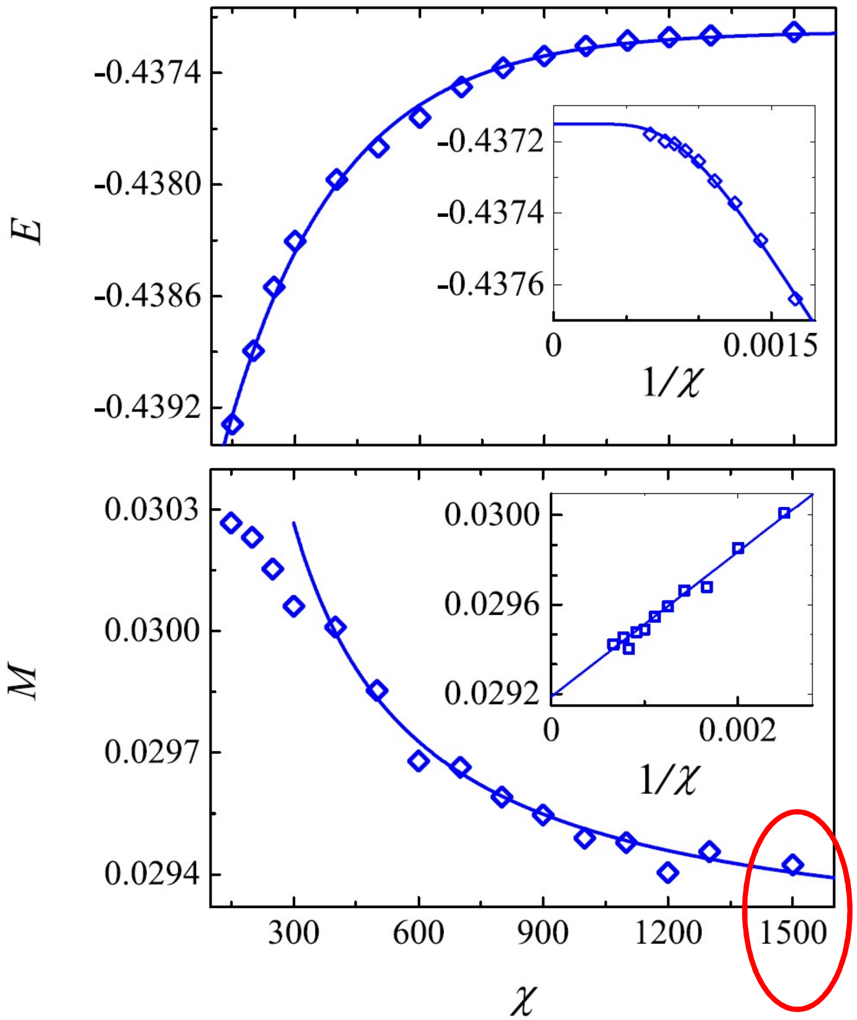
$$|\Psi\rangle = \sum_{\{m\}} \sum_{ijkl...} A_{ia}[m_1]B_{jb}[m_2]C_{kc}[m_3]S_{1,abc}S_{2,ij'k'}...|m_1m_2m_3...\rangle$$



Published Result: with D up to 13

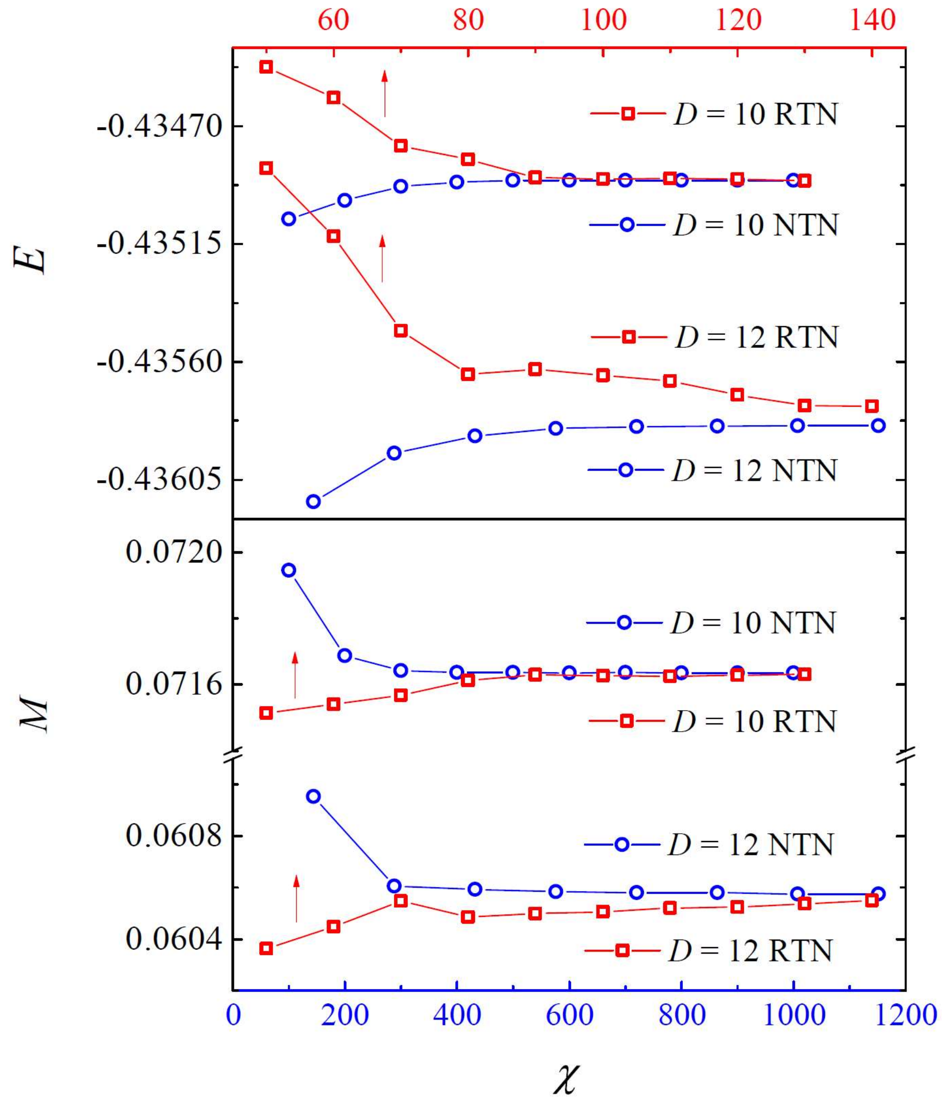


When RTN can not work: D = 24 e.g.



Energy: already exp. converged
Mag: only pow. converged, need to increase χ

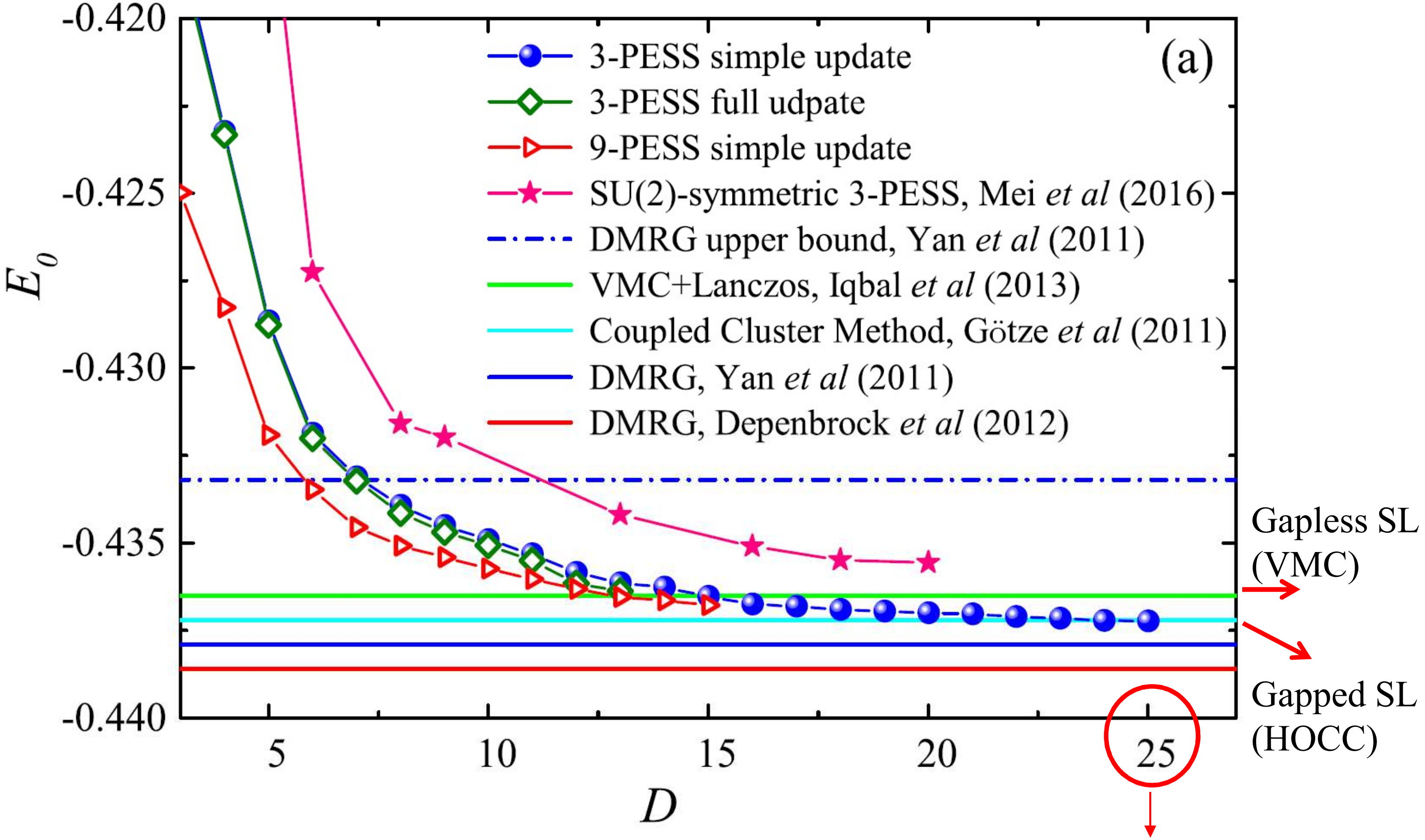
Convergence Test: when RTN can be applied



This is important to ensure the convergence: X~D^2
RTN: X~100
NTN: X~1000

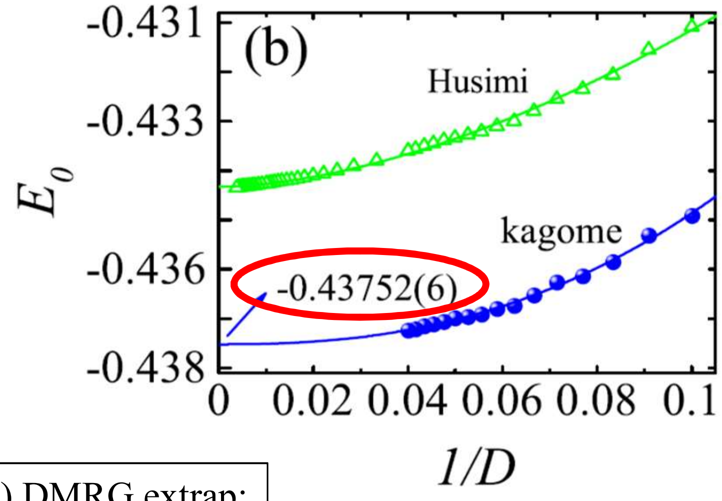
With the help of Nested Tensor Network

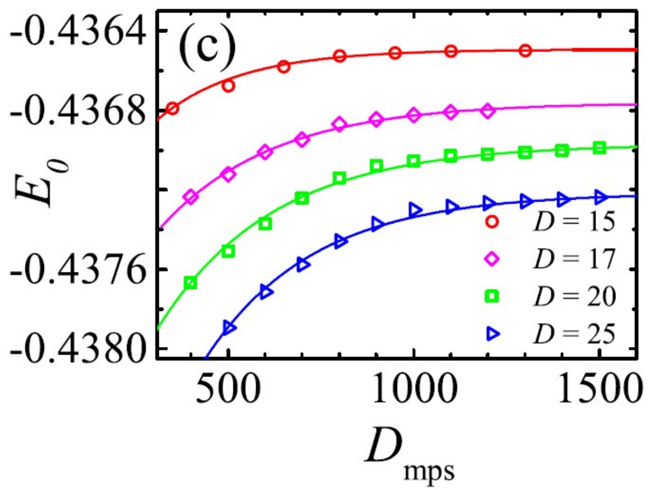
HJLiao, *et al*, PRL 118, 137202 (2017)



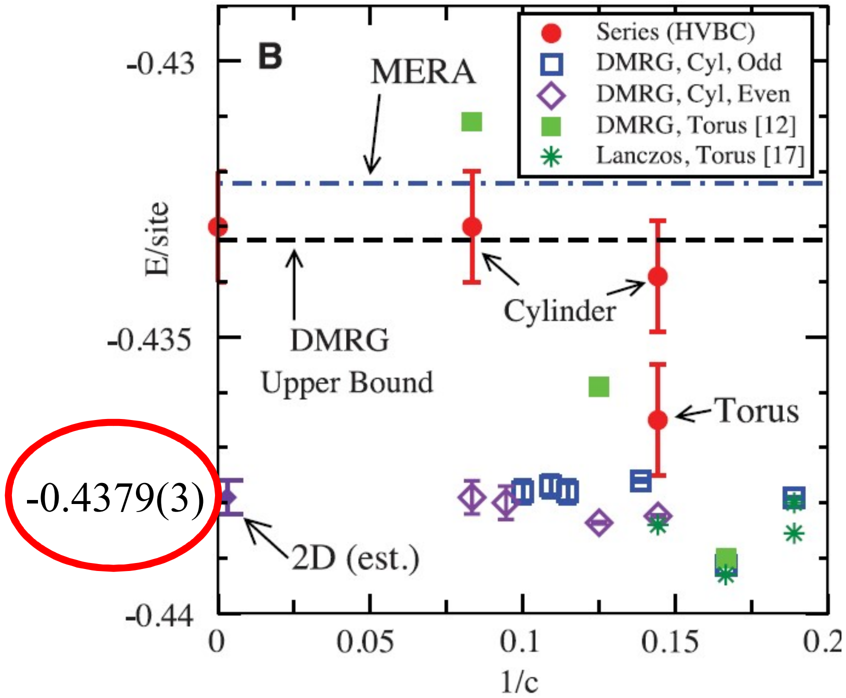
Former memory cost: 1.1T~2.2T

χ = 1500, impossible for RTN

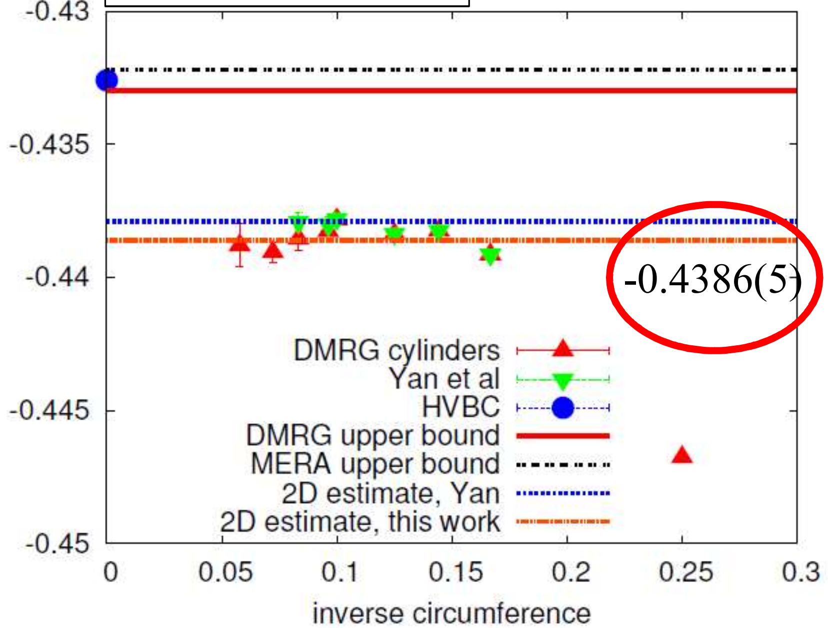




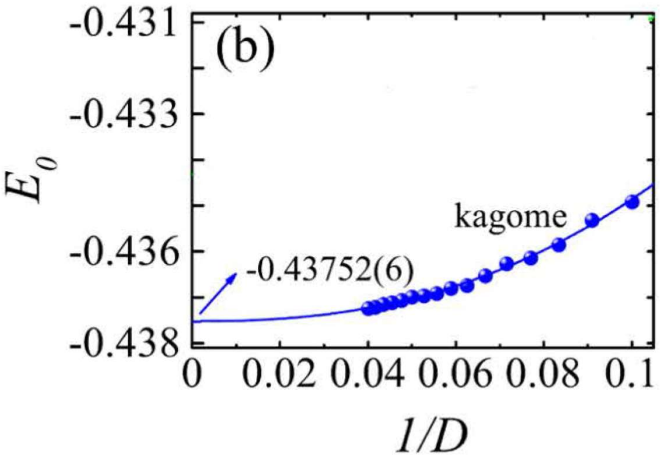
U(1) DMRG extrap:

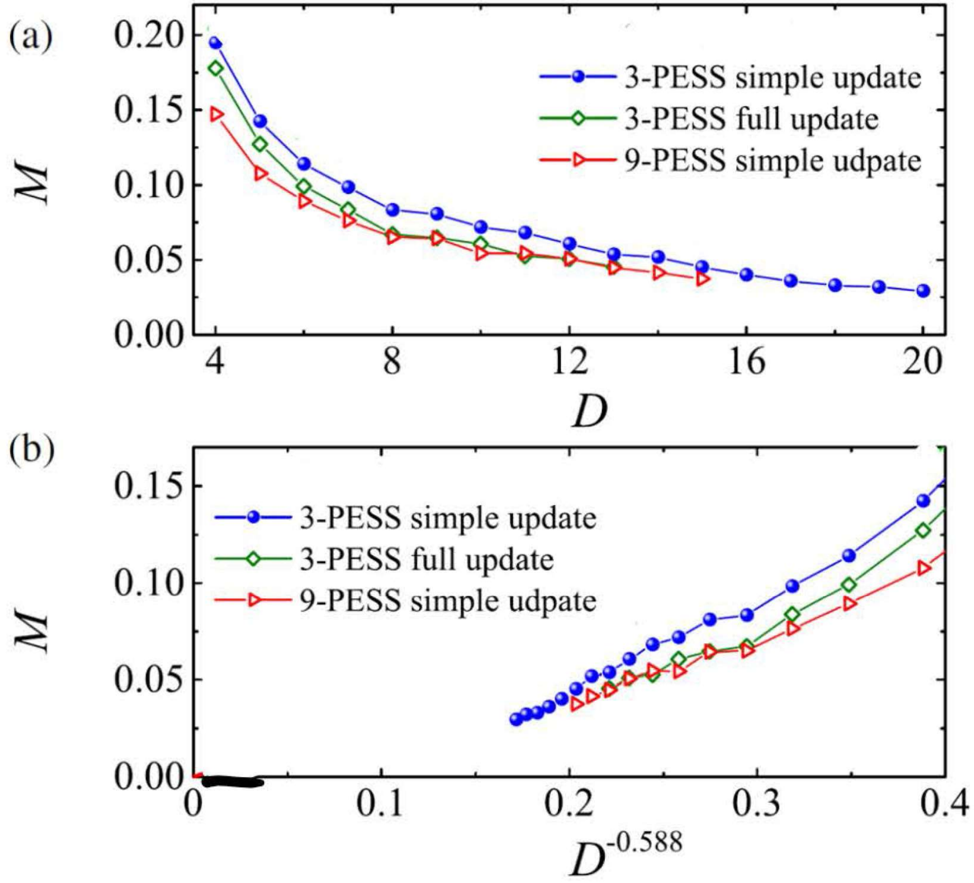


SU(2) DMRG extrap:



Large D tells more: probably to be gapless



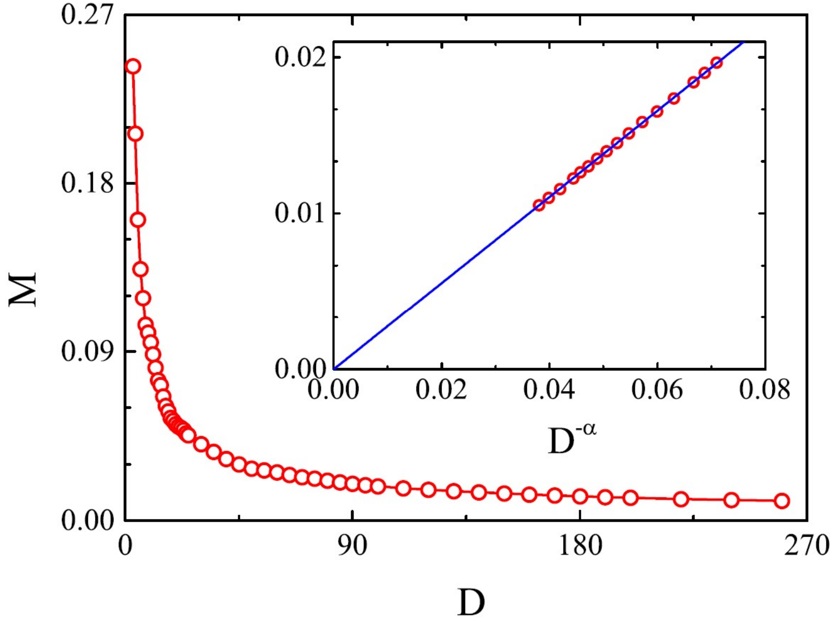


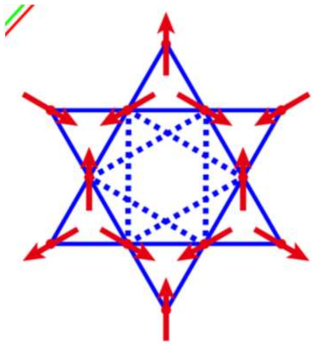
Scaling-Gap relation: (general belief)

Gapless/critical system: polynomial
Gapped system: exponential

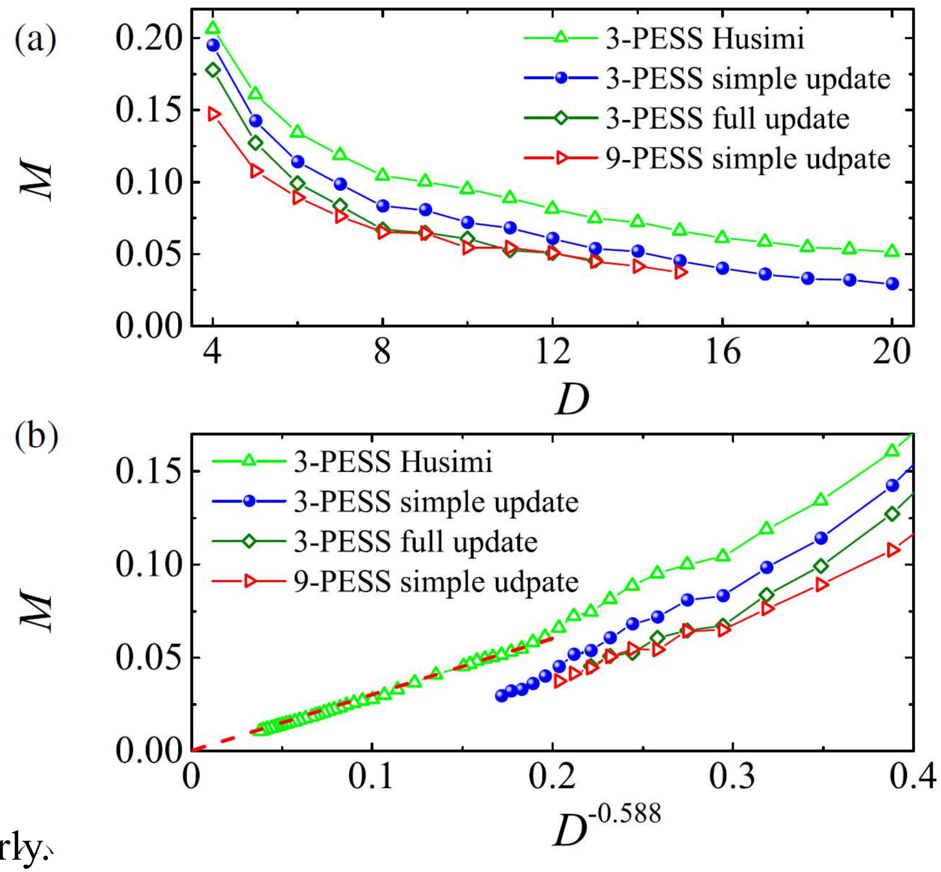
➢ Full update (D~13), large cluster (D~15) does not change too much

Husimi lattice



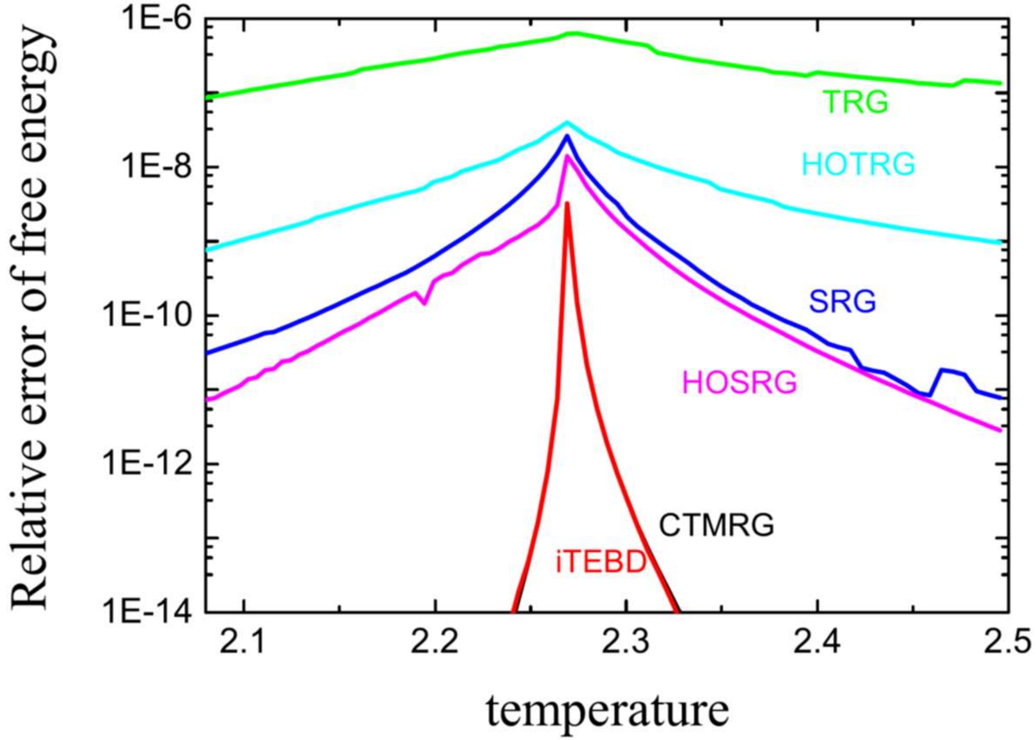


Comparison between Husimi and Kagome



- ✓ At finite D, both of them are ordered, and behaves very similarly.
- ✓ At infinite D limit, Husimi is of no order, i.e., M = 0
- ☐ If Husimi Mag is still upper bound when D approaches infinity, then Kagome **Mag should also vanish**!
- ☐ If there is no finite-D transition, then probably **gapless**.

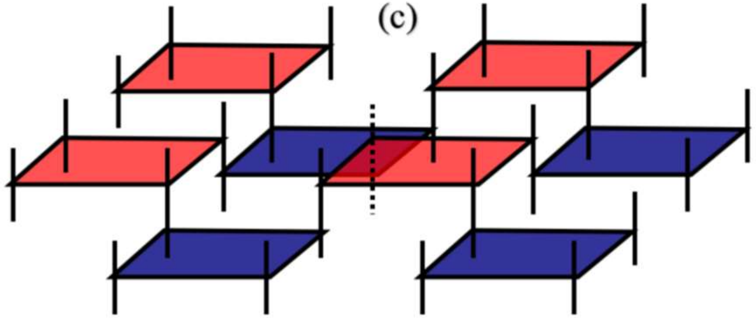
Application 2: Ising model on cubic lattice

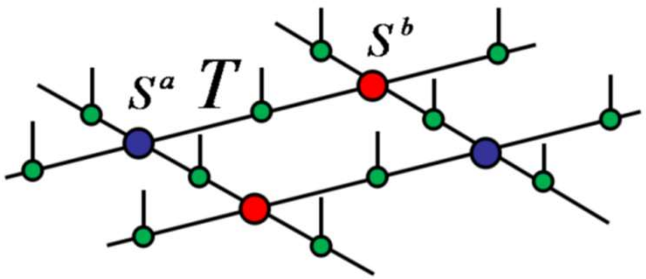


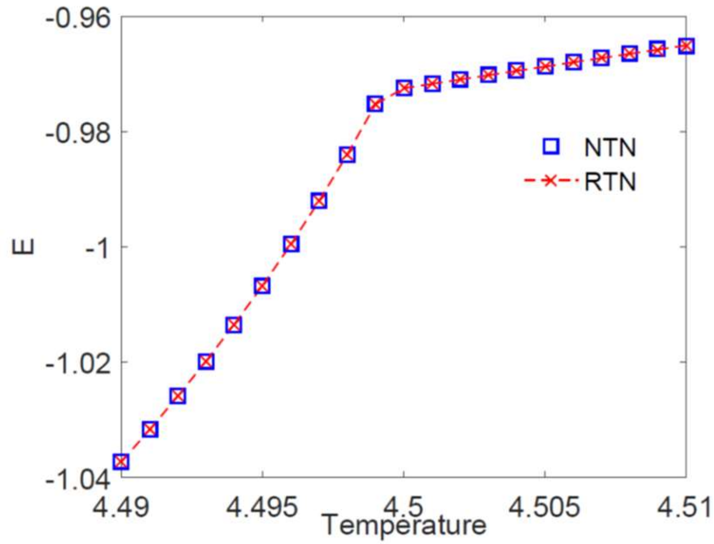
Motivation:

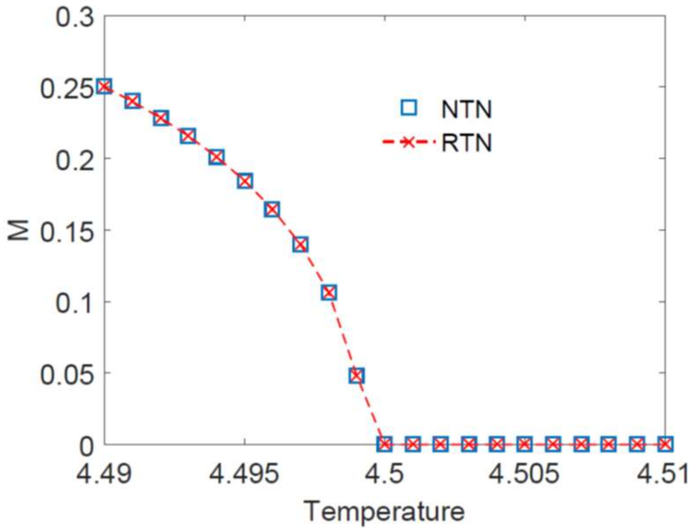
1. HOTRG works well in 3D
   iTEBD should works better
2. Gain insight for quantum lattice model
   (more efficient wavefunction update scheme)

Ising model on square lattice: D = 30


(c)

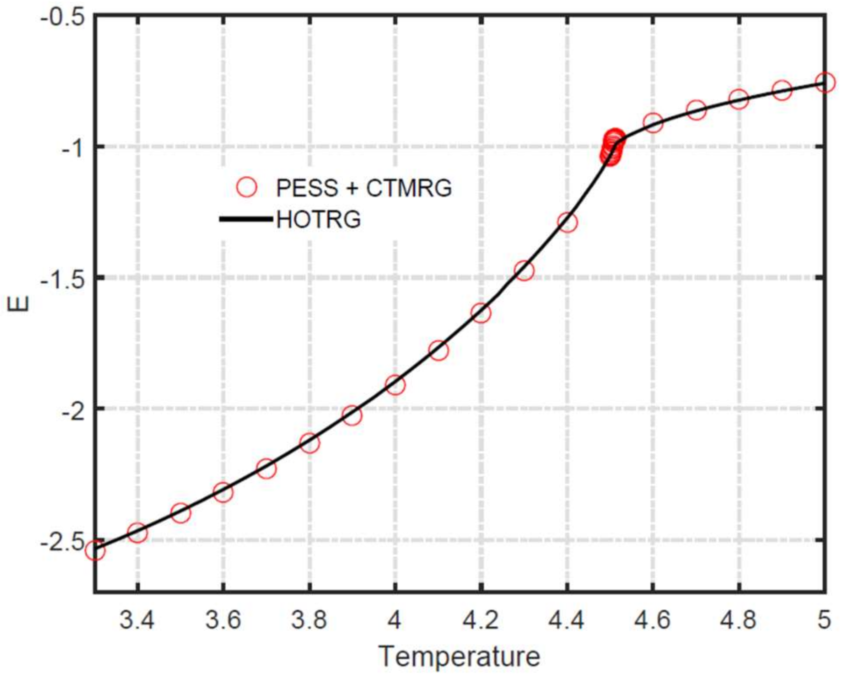

S^a
T
S^b

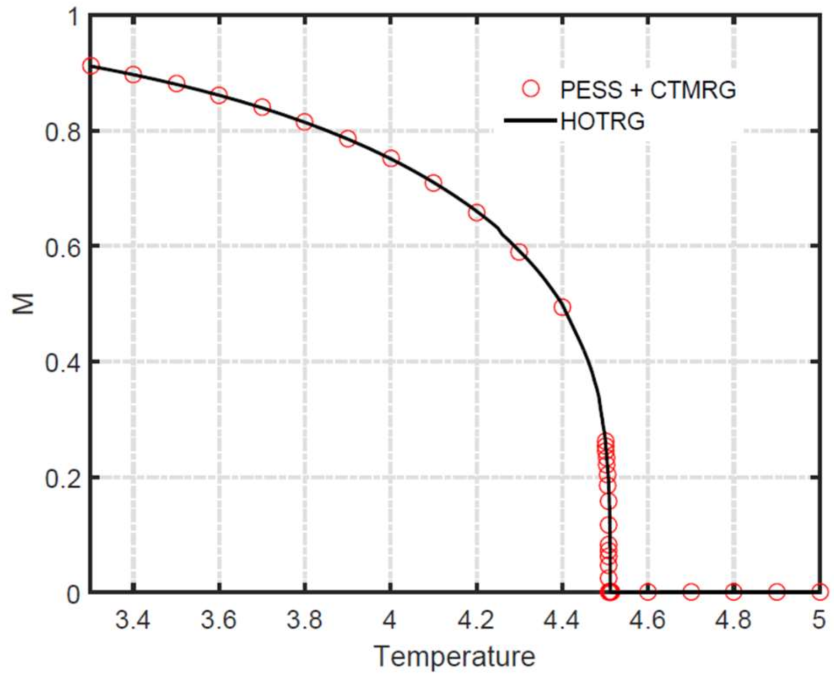

-0.96
-0.98
-1
-1.02
-1.04
E
NTN
RTN
4.49
4.495
4.5
4.505
4.51
Temperature


0.3
0.25
0.2
0.15
0.1
0.05
0
M
NTN
RTN
4.49
4.495
4.5
4.505
4.51
Temperature

D = 4, x = 20

Produce reliable tensor-network state result, probably the 1st time

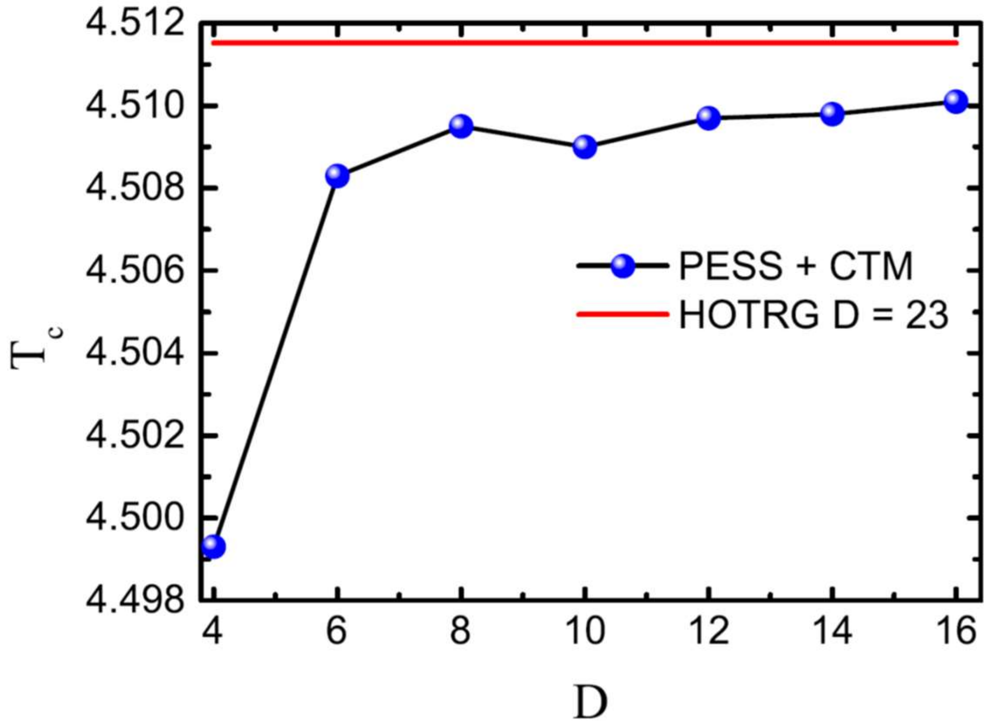




D = 10, x = 160 (guaranteed to be very accurate)

Wavefunction-based method **can** also work well in 3D, if the wf ansatz is properly chosen

Z. Y. Liu, *et al*, in preparation



Method	T_c
TPVA (2001)[38]	4.5393
Series Expansion (2000)[165]	4.511536(21)
Monte Carlo (2010)[164]	4.5115232(17)
HOTRG (2014)[104]	4.51152469(1)
PESS + CTMRG(D = 16, this work)	4.5101(1)

PESS wavewfunction ansatz is important
Can be improved by better update method

Summary

- Calculation of the wavefunction overlap (or inner product) is a central part of TRG for quantum lattice models, and also a main bottleneck of almost all TRG methods.
- **Nested Tensor Network** method aims to solve the problem
  - Idea: change the order of summation over the local tensors
  - Memory: D^2, computation: D^3, accuracy: improved by much larger χ (10 times)
- **Critical 1**: Anti-ferromagnetic Kagome Heisenberg Model: **PESS wf**
  - More reliable grdst Energy: -0.43752(6), D=25, x=1500
  - Finite D: q=0 ordered state, extrapolation: gapless SL E/M converge polynomially (U(1)-Dirac-fermion)
- **Critical 2**: Ising model on cubic lattice: **PESS wf**
  - Produce reliable result for thermodynamic quantities (E, M): This is first!
  - Obtain comparable Tc with other techniques, promising for better update, D=16

Main Collaborators in this talk:

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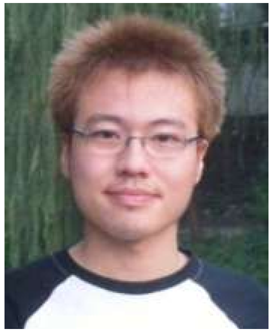
Prof. Bruce Normand
PSI, Switherland

Dr. Haijun Liao
IOPCAS, China



Dr. Jing Chen
IOPCAS, China

Dr. Zhiyuan Liu
ITPCAS, China

Main Refs: ZYXie, H. J. Liao, R. Z. Huang, H. D. Xie, J. Chen, Z. Y. Liu, T. Xiang, **Phys. Rev. B** 96, 045128 (2017)

H. J. Liao, ZYXie, J. Chen, Z. Y. Liu, H. D. Xie, R. Z. Huang, B. Normand, T. Xiang, **Phys. Rev. Lett**. 118, 137202 (2017)

ZYXie, J. Chen, J. F. Yu, X. Kong, B. Normand, T. Xiang, **Phys. Rev. X** 4, 011025 (2014)

Z. Y. Liu, ZYXie, T. Xiang, **to appear**

Thanks!

Next, we calculate the Kitaev model on the honeycomb lattice [52] with an equal amplitude of bond couplings defined by

$$H = \frac{1}{2} \sum_{\langle i,j \rangle_\gamma} \sigma_i^\gamma \sigma_j^\gamma, \quad \gamma = z,x,y \tag{47}$$

where the Ising anisotropy depends on the three different directions of bonds of the honeycomb lattice. Here, σ_i^γ represents the Ising spin at i site with the Ising anisotropy axis in the γ direction. This model is exactly solvable at any lattice size with any kind of the PBC [52]. The exact ground state is a highly frustrated spin liquid and it is gapless in the thermodynamic limit. The infinite PEPS calculation [53] shows good agreement with the exact ground-state energy in the infinite lattice, although the lattice rotational symmetry is artificially broken due to the mapping from honeycomb lattice to brick-wall lattice.

The Lieb-Schultz-Mattis (LSM) theorem [1], and its higher-dimensional generalizations [2,3], state that if a quantum spin system defined on a lattice has odd number of spin-$\frac{1}{2}$ per unit cell, then any local spin Hamiltonian which preserves the spin and translation symmetry cannot have a featureless (gapped and nondegenerate) ground state. This implies that any symmetry-allowed Hamiltonian on the spin Hilbert space defined above can only have the following possible scenarios: (i) its ground state spontaneously breaks either the spin symmetry or the lattice symmetry, hence leads to degenerate ground states and possible gapless Goldstone modes; (ii) it has gapped and degenerate ground states without breaking any symmetry, i.e., its ground state develops a topological order (the second possibility can only happen in two- and higher-dimensional systems); (iii) its ground state has algebraic (power-law) correlation function of physical quantities, and the spectrum is again gapless [this scenario happens most often in one-dimensional (1D) spin systems, while still possible in higher dimensions].

E.H. Lieb, T.D. Schultz, and D.C. Mattis, Ann. Phys. **16**, 407 (1961)

M. B. Hastings, Phys. Rev. B **69,** 104431 (2004) .

Lieb-Schultz-Mattis-Hasting Theorem

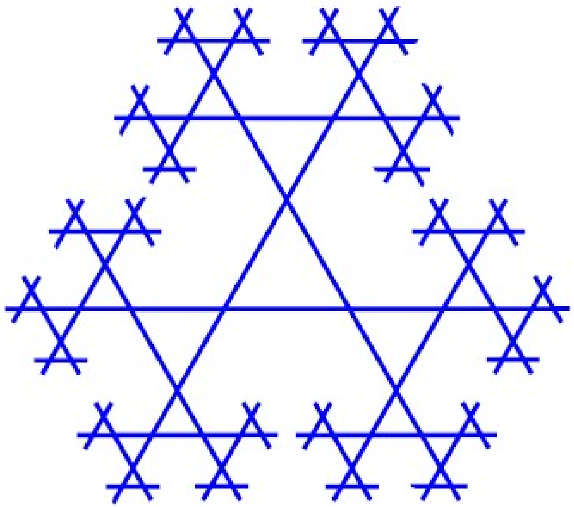
LSMH theorem: SU(2) invariant Hamiltonian with odd-half-integer spin per unit cell, which implies following three cases:

- symmetry-breaking: goldstone mode
- Gaped spin liquid with degenerate ground states
- Gapless spin liquid

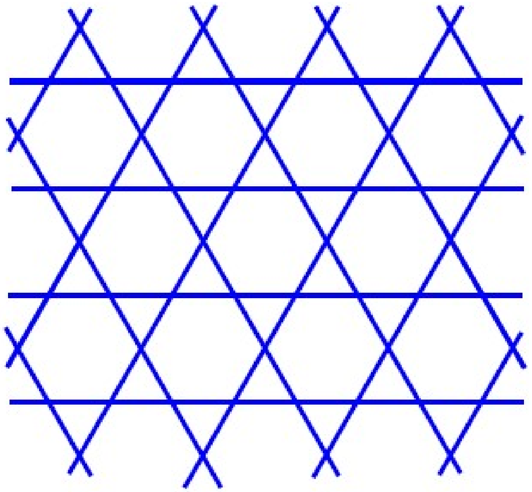
E.H. Lieb, T.D. Schultz, and D.C. Mattis, Ann. Phys. **16**, 407 (1961)

M. B. Hastings, Phys. Rev. B **69,** 104431 (2004) .

Some evidence of the scaling-gap relation



(p=3,q=2) Husimi



Kagome

1D: much evidence
2D: less evidence (due to small D), well believed

Simple argue band not rigorouse at all:
for gapped system

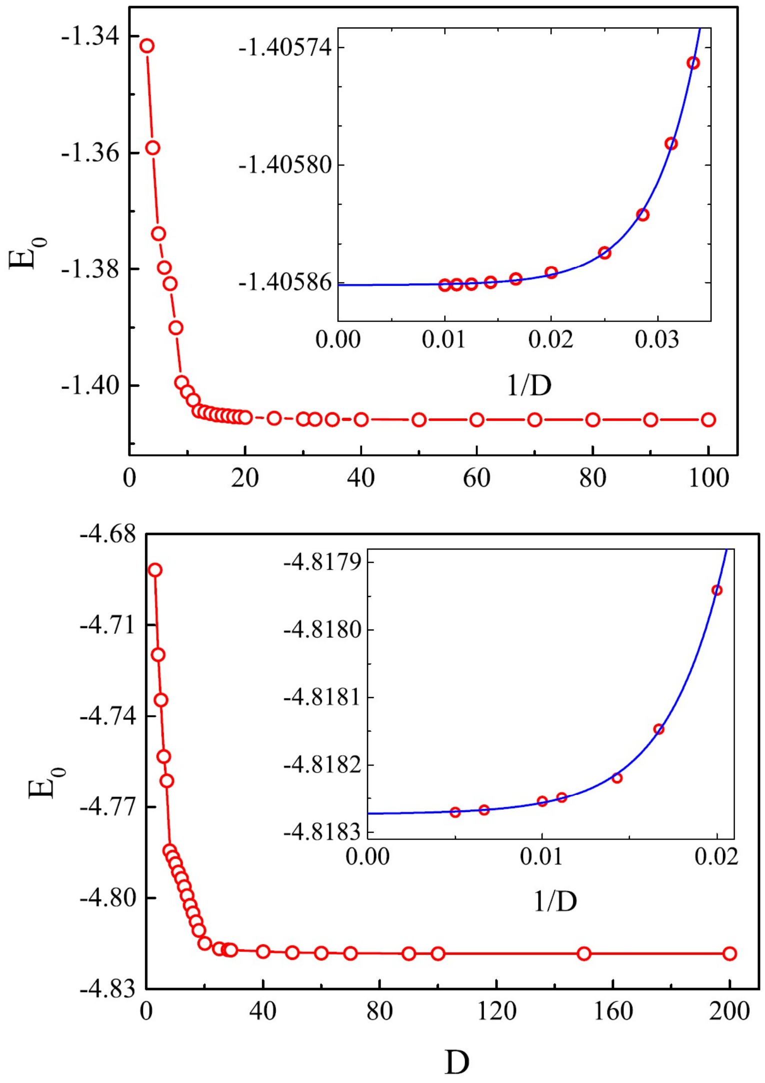
$$S = \alpha L^{d-1}$$
$$S_D = L^{d-1} \log D$$
$$\varepsilon = 1 - S_D/S = 1 - \frac{\log D}{\alpha} \sim \log \frac{1}{D}$$

Husimi:
each bond belongs to a single polygon/loop
much less frustrated, and ***much easier to study***!

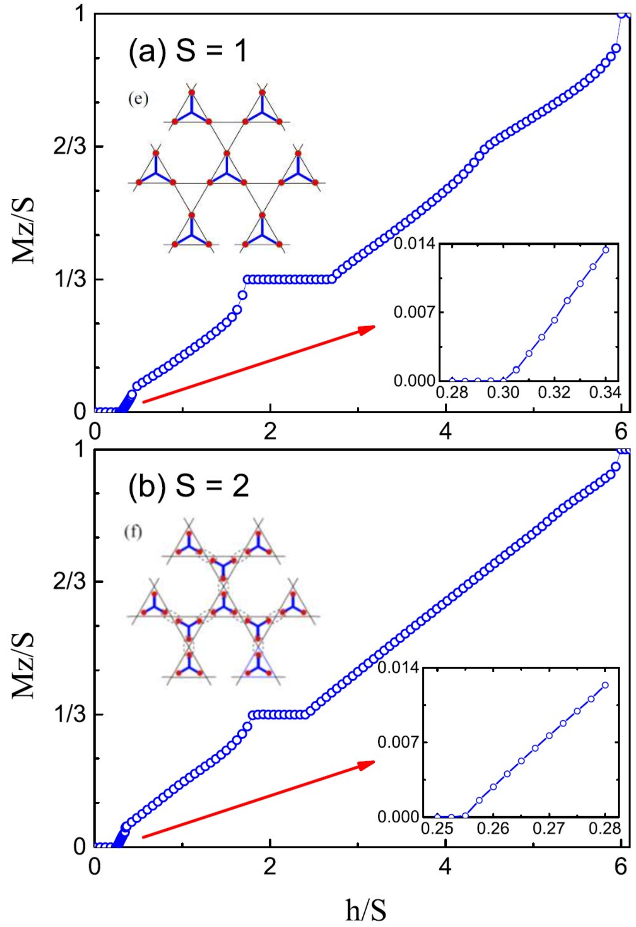
$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i \mathbf{S}_i^z,$$

HJLiao, *et al*, PRB 93, 075154 (2016)

About the scaling-gap relation



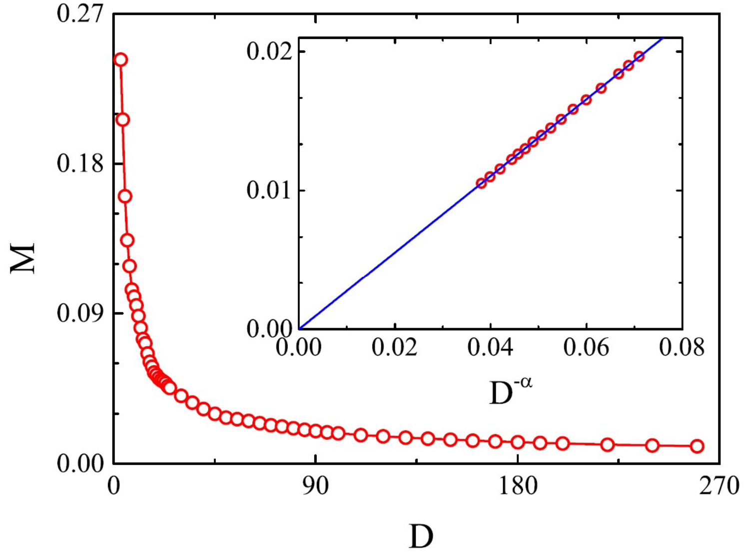
Gapped SSS (trimerized, non-uniform)

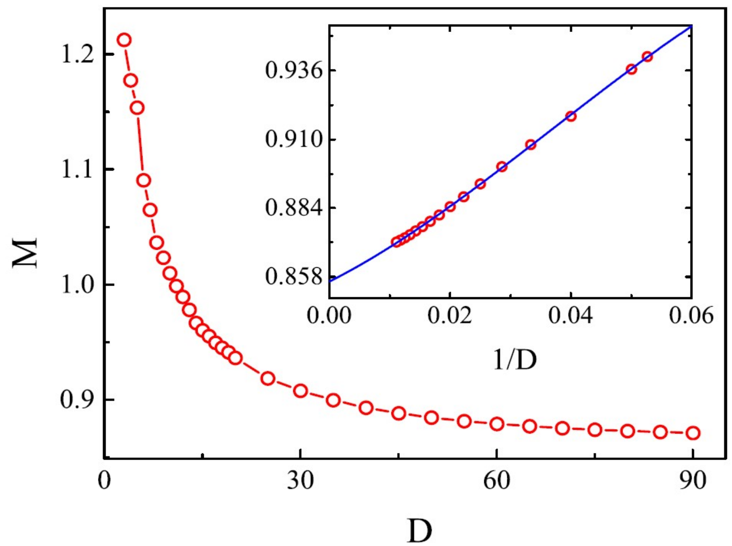


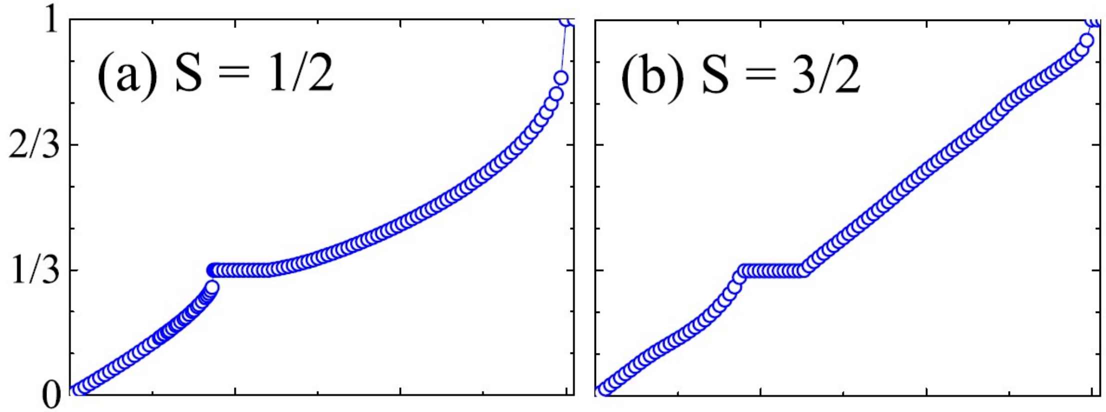
Gapped SSS (uniform)

HJLiao, *et al*, PRB 93, 075154 (2016)

Quasi-2D Example: About the scaling-gap relation





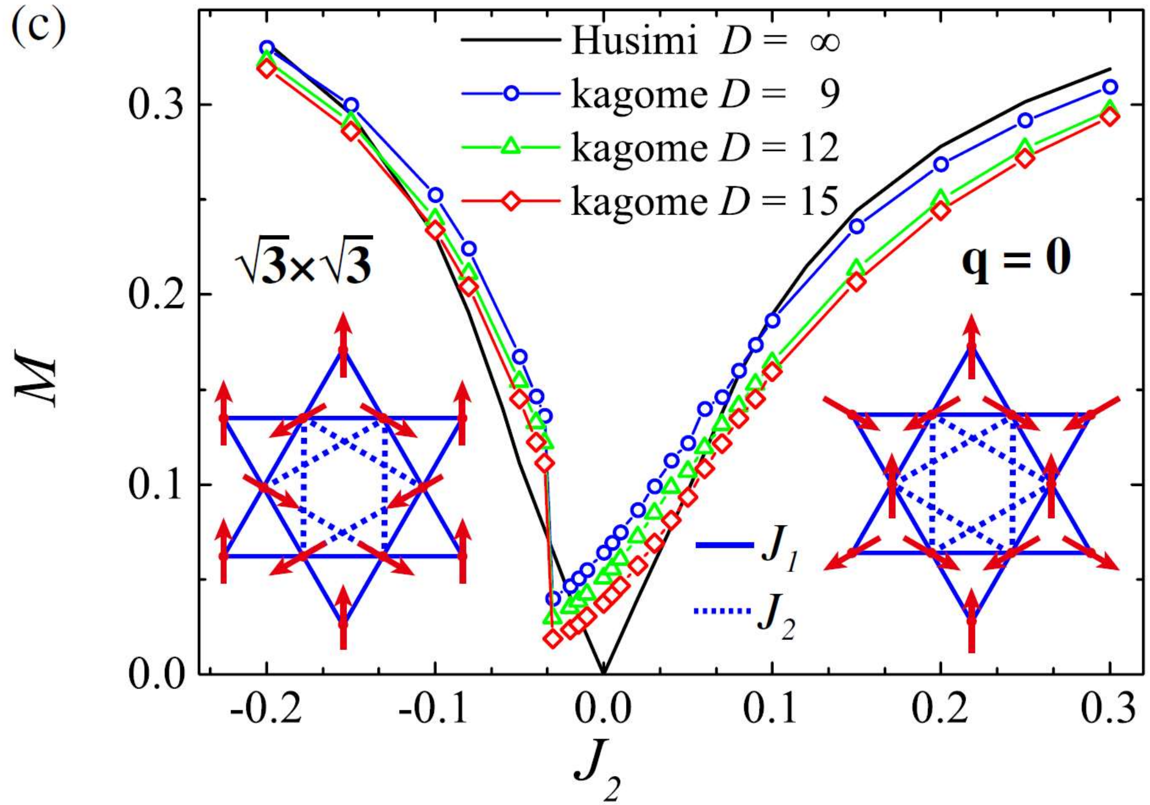


Gapless SL

120° AF Neel Order

Just to mention:
S>2, always ordered
Just like S=3/2(gapless)

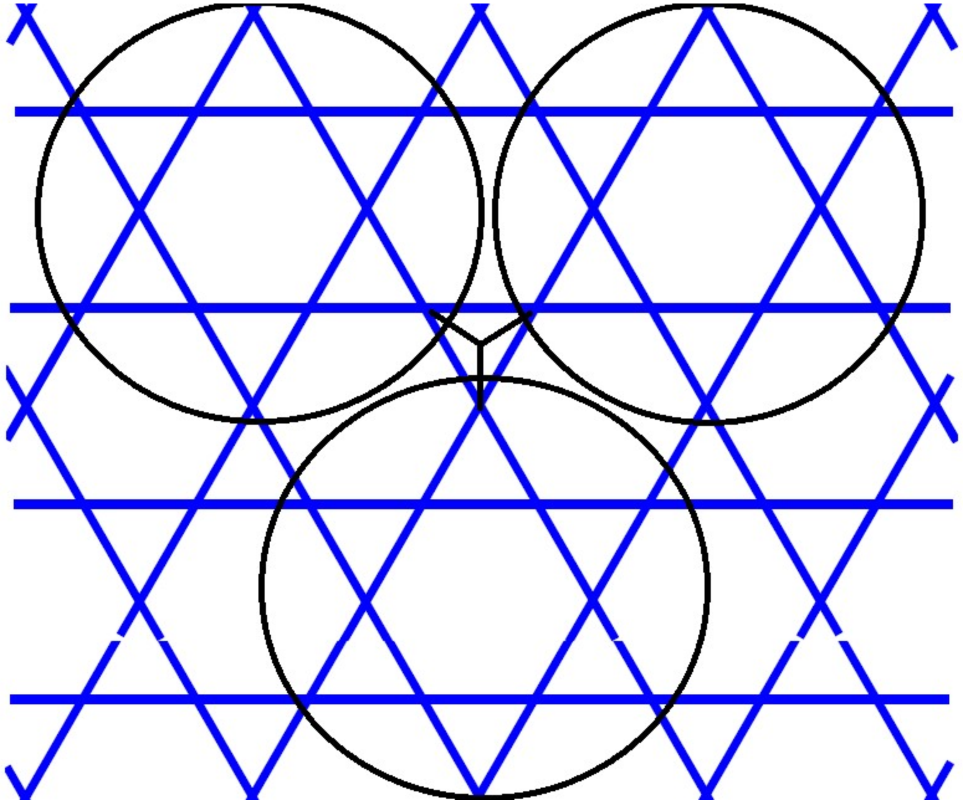
Supplementary Information: simple update with second neighbor coupling J_2



Seems like a stable phase: (-0.03, 0.04)

Unsuitable channel: always (q=0) ordered at finite D!

36-PESS



Fermions with 2D tensor networks

Simulate fermions in 2D?

Before April 2009: **NO!**

Since April 2009: **YES!**

Different formulations:

P. Corboz and G. Vidal, Phys. Rev. B 80, 165129 (2009)

T. Barthel, C. Pineda, and J. Eisert, Phys. Rev. A 80, 042333 (2009)

C. V. Kraus, N. Schuch, F. Verstraete, J. I. Cirac, Phys. Rev. A 81, 052338 (2010)

Q.-Q. Shi, S.-H. Li, J.-H. Zhao, and H.-Q. Zhou, arXiv:0907.5520

P. Corboz, G. Evenbly, F. Verstraete, G. Vidal, Phys. Rev. A 81, 010303(R) (2010)

C. Pineda, T. Barthel, and J. Eisert, Phys. Rev. A 81, 050303(R) (2010)

P. Corboz, R. Orus, B.Bauer, G. Vidal, PRB 81, 165104 (2010)

I. Pizorn, F. Verstraete, Phys. Rev. B 81, 245110 (2010)

Z.-C. Gu, F. Verstraete, X.-G. Wen. arXiv:1004.2563

Classification by entanglement

Entanglement

low	**gapped systems**	band insulators, valence-bond crystals, s-wave superconductors, ...
	gapless systems	Heisenberg model, p-wave superconductors, Dirac Fermions, ...
high	**systems with "1D fermi surface"**	free Fermions, Fermi-liquid type phases, bose-metals?

$$S(L) \sim L \log L$$