如何分析变量关系 (Newton / Lebnitz)

two sets : A and B

maps : $F: \quad A \to B$

$\qquad\qquad a \mapsto F(a)$

$$F(a) \equiv b \in B$$

F : a rule, assigning to each element a of A : $\underline{\text{a}}$ corresponding element b of B
(一个)

A : domain (域)　　　B : codomain (陪域)

$a \in A$ → argument　　　$F(a)$ ↓ image (element)

Differentiation of one-dimensional functions

Definition

Let $f: \mathbb{R} \to \mathbb{R}$

$y \mapsto f(y)$ be a smooth function.

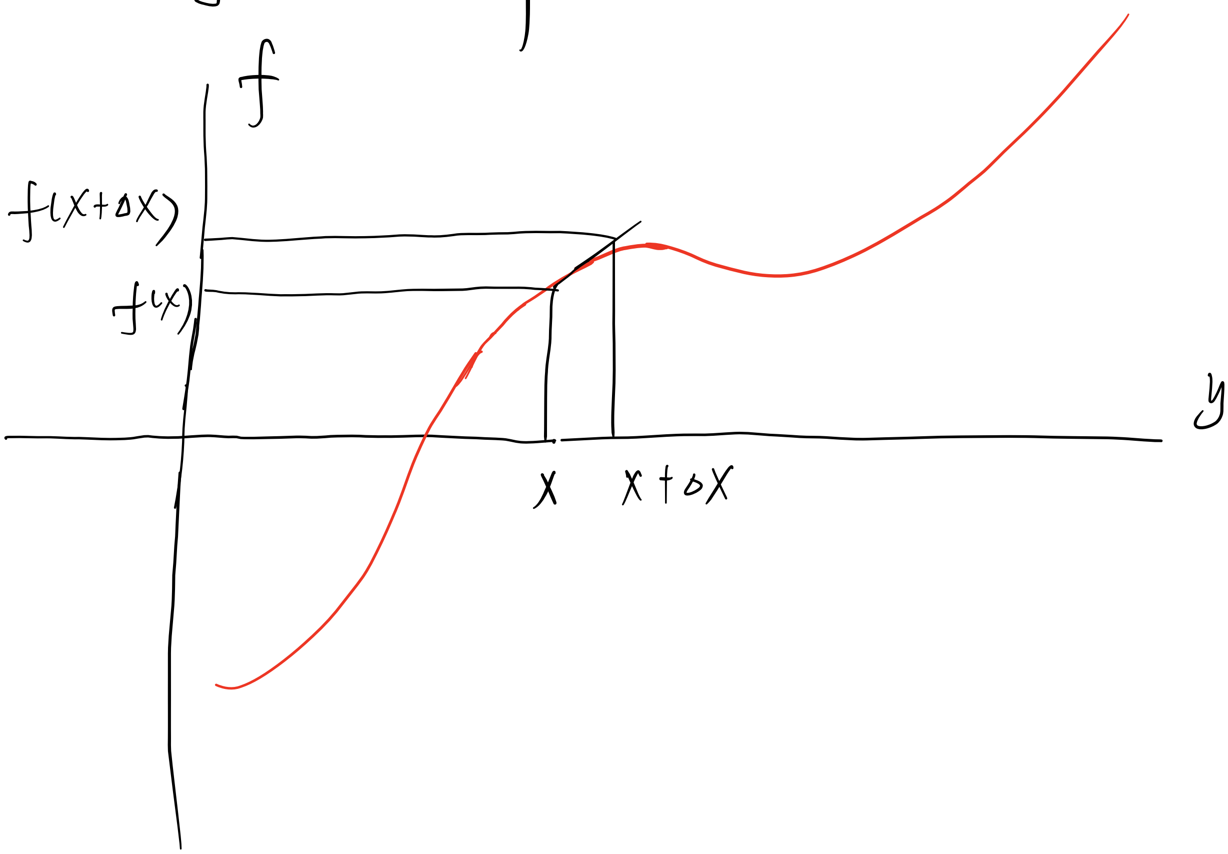
$$f'(x) = \frac{df(x)}{dx} \equiv \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(f(x+\Delta x) - f(x) \right) \quad (I)$$

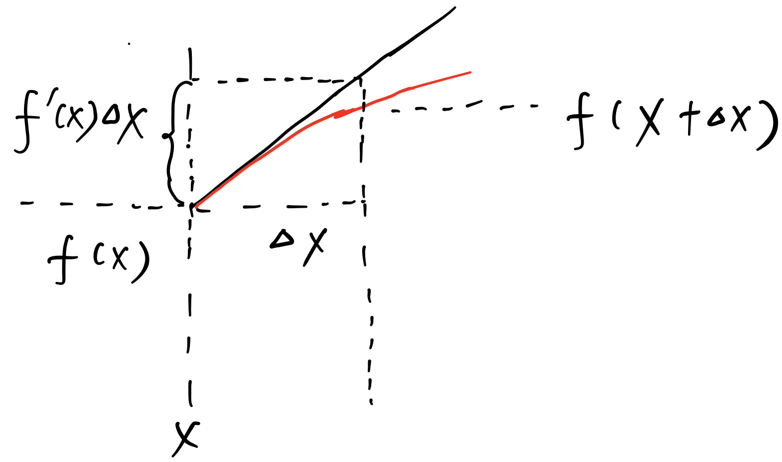
~~derivative of $f(x)$ with respect to x~~

if $f'(x)$ determines the slope of f at x

just ok!

A more powerful interpretation goes as follows:





Δx : small but not infinitesimally small

$$\frac{1}{\Delta x}\left(f(x+\Delta x) - f(x) \right) \simeq f'(x)$$

only in the limit $\Delta x \to 0$

the approximate equality becomes exact

$$f(x+\Delta x) \simeq f(x) + f'(x)\,\Delta x$$

setting $x + \Delta x = y$, i.e.

$$f(y) \simeq f(x) + f'(x)(y-x)$$

y is in the immediate neighborhood of x

$\boxed{\text{All derivatives represent local approximations of functions by linear functions}}$

Keep this point in mind,

no difficulty in understanding

even very fancy derivative operations

E.g. $\sin\phi \to \sin(\phi+\delta)$

$$= \sin\phi \underbrace{\cos\delta}_{=1} + \underbrace{\sin\delta}_{=\delta} \cos\phi$$

$\delta \to 0$

$$= \sin\phi + \delta\cos\phi$$

$$(\sin\phi)' = \frac{d\sin\phi}{d\phi} = \lim_{\delta\to 0}\frac{\sin(\phi+\delta)-\sin\phi}{\delta}$$

(here, $\delta \equiv \Delta\phi$)

$$= \cos\phi$$

课堂练习： $(\cos\phi)' = ?$

$$\left(\frac{1}{x}\right)' = ?$$

$$\Delta f(x) = \frac{1}{x+\Delta x} - \frac{1}{x} = \frac{-\Delta x}{x(x+\Delta x)}$$

$$\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{-\Delta x}{x^2\left(1+\frac{\Delta x}{x}\right)\Delta x}$$

$$= -\frac{1}{x^2}$$

generalization

$\downarrow$ derivative of higher order

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{d^n f(x)}{dx^n} = \frac{d^n}{dx^n} f(x) =$$

$$\underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} \cdots \left(\frac{d}{dx} f(x)\right)\right)\right)}_{n \text{ factors}}$$

(The notation)

$$f^n(x) \equiv \frac{d^n}{dx^n} f(x) \quad (\text{量纲同}[x^n])$$

△ sum rule

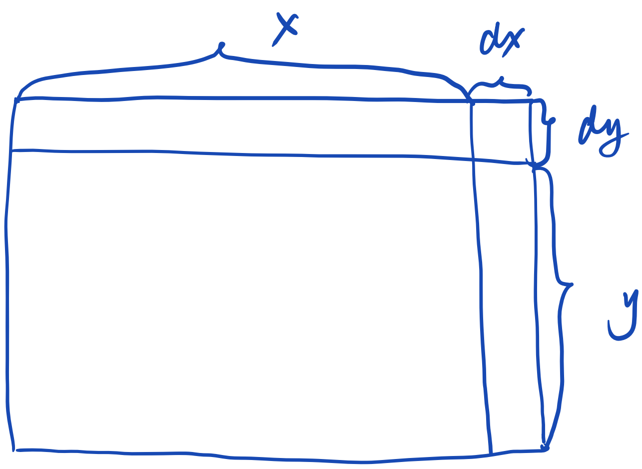
$$(f(x)+g(x))' = f'(x)+g'(x)$$

△ product rule

$f \quad g$

$$\frac{d(fg)}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

e.g.



$$D(xy) = (x+dx)(y+dy) - xy = x\,dy + y\,dx + \underset{(O^2(\varepsilon))}{\underset{\downarrow}{dx\,dy}}$$

$$\xrightarrow{\text{令 } y = x}$$

$$d(x^2) = x\,dx + x\,dx = 2x\,dx$$

$$(x^2)' = \frac{d(x^2)}{dx} = 2x$$

product rule proof:

$$f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)$$

$$= [f(x+\Delta x) - f(x)]\,g(x+\Delta x) + f(x)\,[g(x+\Delta x) - g(x)]$$

then $$\frac{d(fg)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} \lim_{\Delta x \to 0} g(x+\Delta x) + f(x) \lim_{\Delta x \to 0} \frac{\Delta g(x)}{\Delta x}$$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

前提：各自的极限存在

比如 $\cos\left(\frac{1}{x}\right)$ 极限不存在

△ chain rule

$$\frac{d\,(f(g(x)))}{dx} = \left.\frac{df(y)}{dy}\right|_{y=g(x)} \frac{dg(x)}{dx}$$

composition：复合

Given two maps, $F: A \to B$ and $G: B \to C$, the composition of the two is defined by

$$G \circ F : A \to C$$

(PS L3)

Alexand Altland

& Jan von Delft

$$a \mapsto G(F(a))$$

[Two examples:

① $$\frac{d f(ax)}{dx} = a \left.\frac{df(y)}{dy}\right|_{y=ax}$$

② $$\frac{d}{dx}\frac{1}{g(x)} = -\frac{1}{g(x)^2}\frac{dg(x)}{dx}$$

choice: $f(y) = \frac{1}{y}$ $\quad y = g(x)$

如何理解极限：（高阶内容）

1° 一尺之棰，日取其半，万世不竭

《庄子·天下篇》

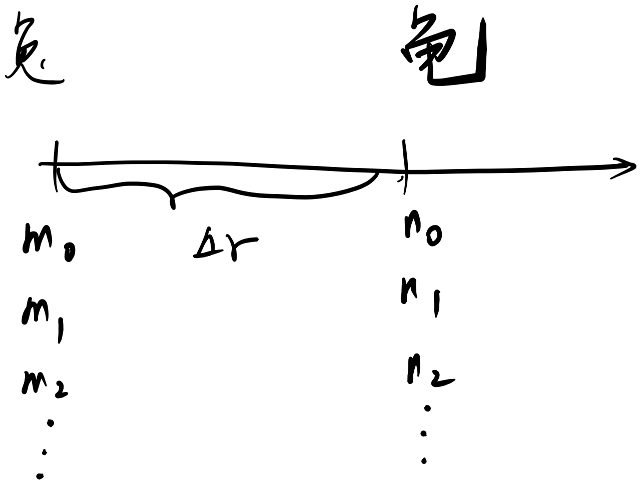
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots$$

$$=1$$ （取完了吗？）

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

2° 龟兔赛跑的悖论

位置$\{m\}$
在时间$\{n\}$上的序列



$n_0-m_0=\Delta r$

$$\Delta t_1=\frac{\Delta r}{v_{兔}}$$

$$\Delta t_2=\frac{\Delta r_1}{v_{兔}}$$

$$\Delta t_3=\frac{\Delta r_2}{v_{兔}}$$

兔子总需要花时间去到达乌龟上一个时间点的位置，永远追不上？

$$\Delta r_1=v_{龟}\,\Delta t_1$$

$$\Delta r_2=v_{龟}\,\Delta t_2$$

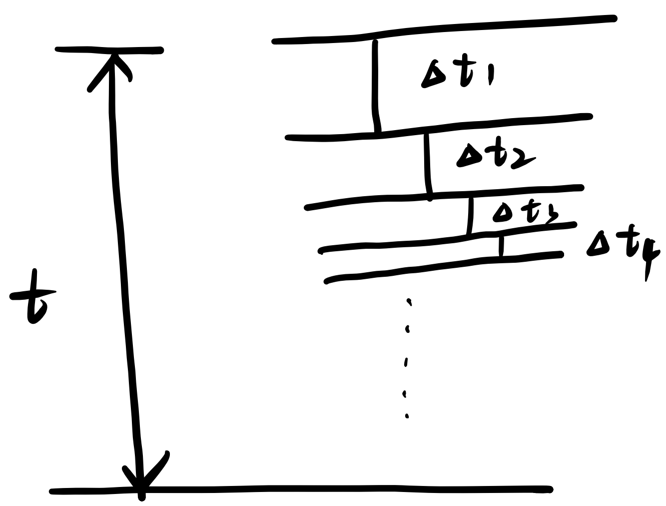
Zeno's paradox

(芝诺时间)

无限步骤 ≠ 无限时间

Δt_n $\quad n \to \infty$ 无限步骤

$$\sum_{n=1}^{\infty} \Delta t_n = t = \frac{\Delta r}{V_{兔} - V_{龟}}$$



无穷数列

$\{\Delta t_n\}$

求和极限

是有限值 t

Derivative of inverse functions (拓展)

$$y = f(x)$$

$$x = f^{-1}(y)$$

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}}$$

$$y = \sin x$$

$$x = \arcsin y$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$= \frac{1}{\cos x} \rightarrow$$ y 是自变量

$$= \frac{1}{\cos(\arcsin y)}$$

$$= 1 / \sqrt{1-y^2}$$

$$\frac{d}{dy}(\arcsin y) = \frac{1}{\sin' x \Big|_{x = \arcsin y}}$$

$$= \frac{1}{\cos(\arcsin y)}$$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin y)}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Newton's method (牛顿方法)

linear approximation (Find $f(x)$)

At $x = a$ $\quad \frac{df}{dx} = f'(a) \simeq \frac{f(x) - f(a)}{x - a}$

$$f(x) \simeq f(a) + (x-a) f'(a)$$

Slove $f(x) = 0$

$$x - a \simeq \frac{f(x) - f(a)}{f'(a)} = - \frac{f(a)}{f'(a)}$$

eg.

Find $\sqrt{9.06} = ?$

set $x = \sqrt{9.06} \Rightarrow x^2 = 9.06 \Rightarrow$

$$x^2 - 9.06 = 0$$

solve $f(x) = x^2 - 9.06 = 0$

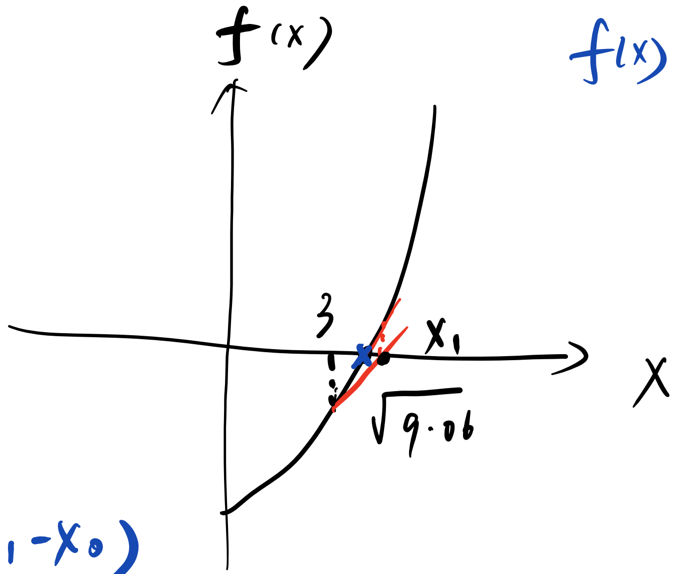
$$x = ?$$

Let us start from $x_0 = 3$

1° 初始估算点… $x_0 = 3$, $f(x_0) = -0.06$

$$f'(x) = 2x \qquad f'(x_0) = 2 \times 3 = 6$$

$f(x) = x^2 - 9.06$



(x_1 : 线性近似与 x 轴交点)

$\therefore f(x_1) = 0$

$$f(x_1) - f(x_0) \approx f'(x_0)(x_1 - x_0)$$

$x_0 \to x_1$

$$x_1 - x_0 = \frac{-f(a)}{f'(a)} = \frac{0.06}{6}$$

$x_1 = 3.01$ (更好的估算点) 相较于 x_0

$$(3.01)^2 = 9.0601$$

2° $x_1 = 3.01$

$$x_2 - x_1 = \frac{-f(3.01)}{2 \times 3.01} = \frac{-0.0001}{6.02}$$

$x_1 \to x_2$

$x_2 = 3.00998$

$\cdots \to x_n$

[后记] 从数集出发，一元函数是从数集到数集的映射，二元函数是从两个数集的笛卡尔集到数集的映射

微分 dx 是个函数

导数 $f'(x_0) = \lim_{\delta \to 0} \frac{f(x_0+\delta)-f(x_0)}{\delta}$

是一个数

如果 $f'(x_0)$ 存在，for $\forall x_0 \in \{x\}$

$f'(x)$ 是导函数：关于自变量 x 的一元函数

如果 $f(x)$ 可导，那么

$y = f(x)$ 的微分 $dy = df(x) = f'(x)dx$

这里的 dy 是二元函数，一个自变量是 x，

一个自变量是 dx（dx 仅仅记号特别）

可视为 $df(x) = g(x, w) = f'(x)\,w$

$(w \equiv dx)$

所谓微分 dy 是关于 dx 的线性函数。

考虑 x， dy 是自变量 x 和自变量 dx 的二元函数

d：微分算子

$$f(x) \xrightarrow{d} df(x) = f'(x)\,dx$$

可导一元函数　　　　二元函数

术语：

微分　differential　adj & n

积分　integral

导数　derivative

y对x的导数

$$\underline{dy} = y'(x)\,\underline{dx}$$

y的微分　　x的微分

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