

Oscillation

$$\textcircled{1} \quad m \ddot{x} + kx = 0 \quad \Rightarrow$$

$$\ddot{x} + \omega_0^2 x = 0$$

$$\textcircled{2} \quad \ddot{x} + \omega_0^2 x = F \cos \omega t$$

$$\textcircled{2}' \quad \ddot{x} + \omega_0^2 x + \gamma \dot{x} = 0$$

$$\textcircled{3} \quad \ddot{x} + \omega_0^2 x + \gamma \dot{x} = F \cos \omega t$$

↓

in general

$$\ddot{x} + \omega_0^2 x + \gamma \dot{x} = F(t)$$

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{i\omega_n t}$$

$$\omega_n = \frac{2\pi n}{T} \quad (n \in \mathbb{Z})$$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = -1, \quad e^{i2\pi} = 1$$

e^{ix} 比 $\cos x$ 求导运算方便

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F \cos \omega t$$

怎么解?

↑

(m 已吸收到 F)

$$\frac{d^2x_r}{dt^2} + \omega_0^2 x_r = F \cos \omega t \quad (1)$$

$$\frac{d^2x_i}{dt^2} + \omega_0^2 x_i = F \sin \omega t \quad (2)$$

$$x = x_r + ix_i$$

$$(1) + (2) \times i \Rightarrow$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F e^{i\omega t}$$

分析：只能含有 X 的 0 或 1 次幂，
否则实数部将混和。

$$\frac{d(e^{i\omega t})}{dt} = i\omega e^{i\omega t}$$

$$\frac{d^2(e^{i\omega t})}{dt^2} = (i\omega)^2 e^{i\omega t}$$

$$\text{令 } X = x_0 e^{i\omega t}$$

$$x_0 (i\omega)^2 e^{i\omega t} + \omega_0^2 x_0 e^{i\omega t} = F e^{i\omega t}$$

$$x_0 = \frac{F}{\omega_0^2 - \omega^2}$$

$\omega_0 < \omega$, 相差为 π

$\omega_0 > \omega$, 相差为 0

$$|x_0| = c |F|$$

C: factor

$\omega \sim \omega_0$, 极强响应

理想驱动 \rightarrow 现实阻尼

$$\frac{d^2x}{dt^2} + \boxed{\gamma \frac{dx}{dt}} + \omega_0^2 x = F \cos \omega t$$

\downarrow friction

$$[(i\omega)^2 x_0 + \gamma(i\omega) x_0 + \omega_0^2 x_0] e^{i\omega t} = F e^{i\omega t}$$

$$x_0 = \frac{F}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$\text{令 } R = \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$x_0 = R F$$

F 是实数, 这说明 x_0 与 F 有相位之差

$$R = \rho e^{i\theta}$$

$$\frac{1}{R} = \frac{1}{\rho} e^{-i\theta}$$

ρ, θ : 响应的大小和相移

* 插入基本知识: 设扰动

$$(X + i\gamma) R$$

$$r e^{i\phi} \rho e^{i\theta}$$

模长 : $r \rightarrow \rho r$

转角 : $\phi \rightarrow \phi + \theta$

here,

$$\rho^2 = \frac{1}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

(*)

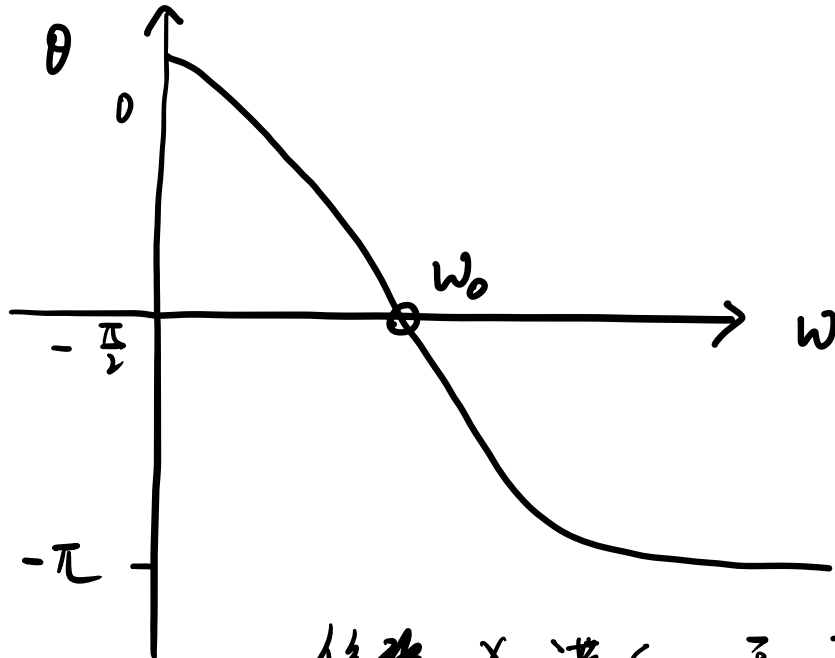
$$\tan \theta = -\gamma \omega / \omega_0^2 - \omega^2$$

$$\text{Im}(R) \sim -\gamma \omega < 0 \quad (\text{三, 四象限})$$

$$\text{Re}(R) \sim \omega_0^2 - \omega^2$$

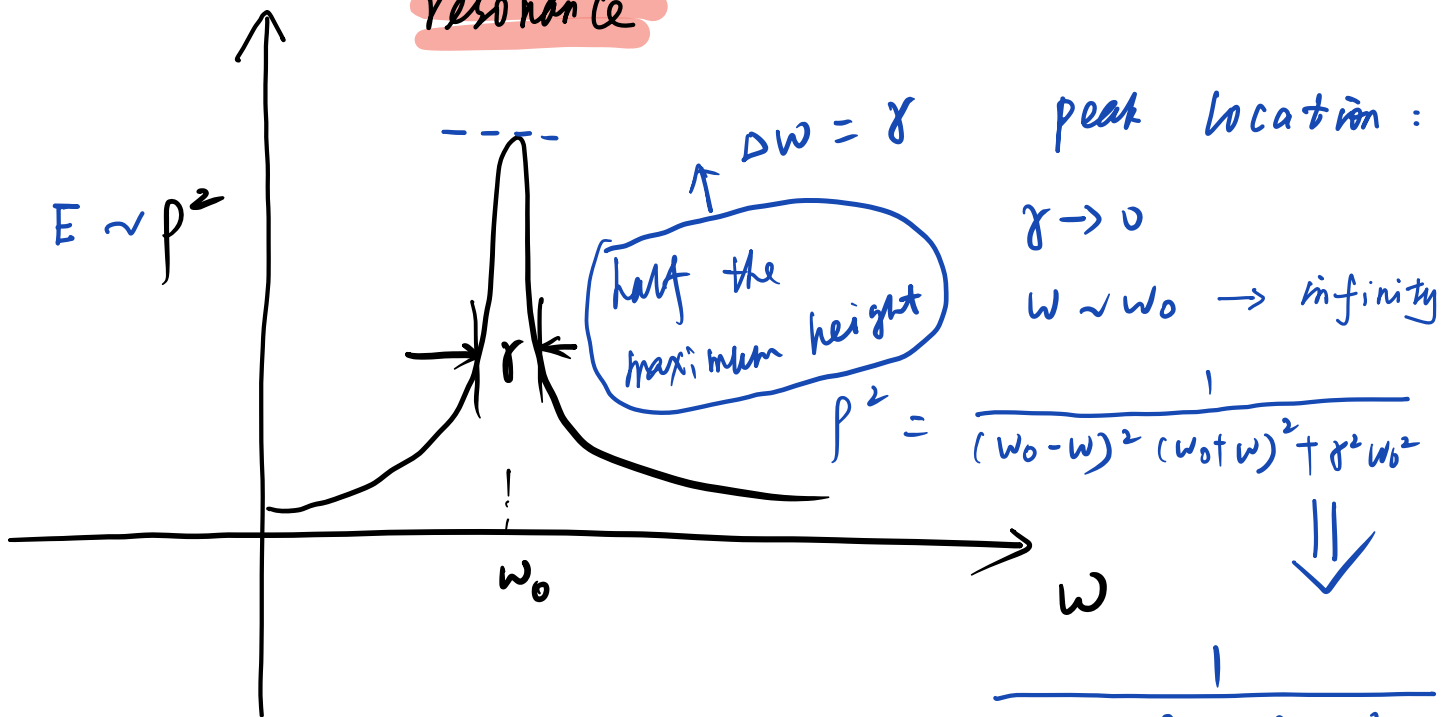


θ 是负数



位移 X 滞后于 F

Resonance



$$p^2 = \frac{1}{(\omega_0 - \omega)^2 (\omega + \omega_0)^2 + \gamma^2 \omega^2}$$

$$\Downarrow$$

$$\frac{1}{(\omega_0 - \omega)^2 4\omega_0^2 + \gamma^2 \omega_0^2}$$

$$R = \omega_0^2 - \omega^2 + i\gamma\omega$$

$$\approx 2\omega_0 (\omega_0 - \omega + i\gamma/2)$$

(if $\gamma \ll \omega_0$, $\omega \approx \omega_0$)

$$= \frac{1}{4\omega_0^2} \frac{1}{[(\omega_0 - \omega)^2 + \gamma^2/4]}$$

$$Q = \omega_0 / \gamma$$

γ 很小, $\omega = \omega_0$ 附近, "共振"峰

如果 go to details.

Where is the maximum? $\omega = \omega_0$?

believe in math!

take the derivative with respect
to ω for eqn. (*)

$$\begin{aligned} \text{分母求导} &= 2(\omega^2 - \omega_0^2) 2\omega \\ (\omega_0^2 - \omega^2)^2 &+ \gamma^2 2\omega = 0 \\ &= (\omega^2 - \omega_0^2)^2 \end{aligned}$$

$$\omega^2 = \omega_0^2 \pm \frac{1}{2} \gamma^2$$

(此式仅对能成立的情况成立)

$$\begin{aligned} \text{代回 } p^2, \text{ 分母} &= \left(\frac{1}{2} \gamma^2\right)^2 + \gamma^2 \left(\omega_0^2 \pm \frac{1}{2} \gamma^2\right) \\ &= \omega_0^2 \gamma^2 + \frac{3}{4} \gamma^4, \quad \omega_0^2 \gamma^2 - \frac{1}{4} \gamma^4 \end{aligned}$$

回到问题本身

$$\theta \in (-\pi, 0]$$

$$x = x_0 e^{i\omega t}$$

$$x = \operatorname{Re}(pF e^{i\omega t} e^{-i|\theta|})$$

"0" for

" $\omega = 0$ "

$$= pF \cos(\omega t - |\theta|)$$

Note: 到目前为止, 我们的根本没使用
初始条件, 何处安放?

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

可以把满足这个方程的 x 补进去

$$\text{在 } D \equiv \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2$$

下 仅仅是补 "0" 而已

求解一下 ~~这个~~ 有阻尼无驱动的情况：

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

$$X = x_0 e^{i\omega t}$$

$$(-\omega^2 + i\omega\gamma + \omega_0^2) x_0 e^{i\omega t} = 0$$

求解一元二次方程

$$\omega^2 - i\omega\gamma - \omega_0^2 = 0$$

$$\omega = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

$$\downarrow \gamma = 2\beta$$

$$\omega = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

$$\omega_0 > \beta$$

under-damping

$$\omega_+ = \sqrt{\omega_0^2 - \beta^2} + i\beta \quad (\equiv \omega_p)$$

$$\omega_- = -\sqrt{\omega_0^2 - \beta^2} + i\beta$$

$$e^{i\omega t} = e^{-\beta t} e^{\pm i\omega_p t}$$

$$X = A e^{i\omega_+ t} + B e^{i\omega_- t}$$

$$X(0) = X_0, \quad V(0) = 0 \quad (\text{initial condition})$$

$$X(0) = A + B = X_0$$

$$V(0) = i\omega_+ A e^{i\omega_+ t} + i\omega_- B e^{i\omega_- t} = 0$$

$$A(\omega_p + i\beta) + B(-\omega_p + i\beta) = 0$$

$$\omega_0 A e^{i\theta} + \omega_0 B e^{i(\pi - \theta)} = 0$$

$$A e^{i\theta} = B e^{-i\theta}$$

$$\Rightarrow |A| = |B|$$

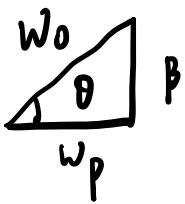
联立

$$A + B = X_0 \Rightarrow A = B^*$$

$$\Downarrow \operatorname{Re}(A) = \operatorname{Re}(B) = \frac{X_0}{2}$$

$$A = \frac{X_0}{1 + e^{2i\theta}} = \frac{X_0 e^{-i\theta}}{2 \cos \theta}$$

$$\downarrow C = \frac{X_0}{2} / \cos \theta$$



则有 $x = e^{-\beta t} (c e^{-i\theta} e^{i\omega_p t} + c e^{+i\theta} e^{-i\omega_p t})$

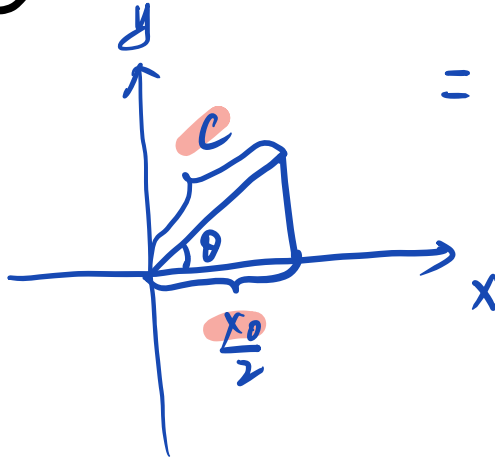
$$= c e^{-\beta t} [2 \cos(\omega_p t - \theta)]$$

$$= \frac{x_0}{\cos\theta} e^{-\beta t} \cos(\omega_p t - \theta)$$

$$\sin\theta = \beta / \omega_0$$

滞后效应

衰减型
振荡



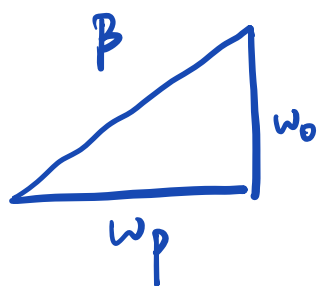
turn to over-damping

$$\omega_0 < \beta$$

$$\omega_p = \sqrt{\beta^2 - \omega_0^2}$$

$$\omega_{\pm} = i\beta \pm i\omega_p$$

$$x = A e^{i\omega_+ t} + B e^{i\omega_- t}$$



$$= A e^{-\beta t} e^{-\omega_p t} + B e^{-\beta t} e^{\omega_p t}$$

$$\beta > \omega_p$$

两种不同程度复减的线性叠加

$$X_0 = X(t=0) = A + B$$

$$V_0 = \frac{dx}{dt} \Big|_{t=0} = i\omega_+ A + i\omega_- B = 0$$

$$A(\beta + \omega_p) + B(\beta - \omega_p) = 0$$

$$\beta(A + B) + \omega_p(A - B) = 0$$

$$\beta X_0 / \omega_p = B - A$$

$$A = \frac{X_0}{2} (1 - \beta / \omega_p), \quad B = \frac{X_0}{2} (1 + \beta / \omega_p)$$

$$A < 0, \quad B > 0, \quad \text{且有 } B > |A|$$

→ critical damping

$$\omega_p = 0$$

$e^{-\beta t}$ is the only solution?

limit approaching!

it can be started from

either side of $\omega_p = 0$

Furthermore, $A e^{-\beta t}$

can not satisfy

$$\begin{cases} X(t=0) = X_0 \\ V(t=0) \neq 0 \end{cases}$$

