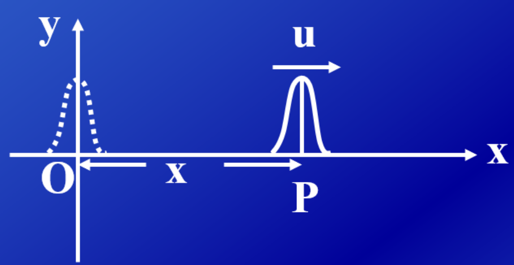
What is a traveling wave ?

Traveling wave



O: $y = f(t)$

P: $y = f(t - \frac{x}{u})$

Wave propagates the “SHAPE” with speed u

$\frac{\partial y}{\partial t}$: vibration $\qquad \frac{\partial y}{\partial x}$: shape

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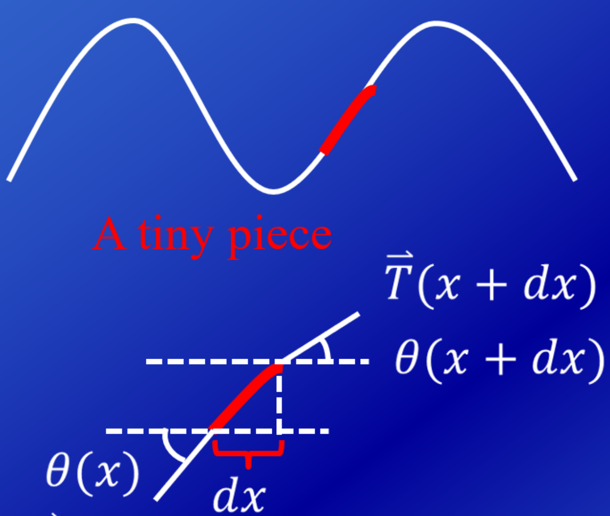
旅行的意义

Wave equation from equation of motion



$\vec{T}(x)$

Newton’ 2nd law

X-direction:

$$T(x)\cos\theta(x) = T(x + dx)\cos\theta(x + dx)$$

Denote $T_x = T$ $\searrow$ constant

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Y-direction:

$$T_y(x+dx) - T_y(x) = (\rho dx)\frac{\partial^2 y}{\partial t^2}$$

$$T_y = T_x \tan\theta = T\frac{\partial y}{\partial x}$$

$$T\frac{\partial^2 y}{\partial x^2}dx = (\rho dx)\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \quad \Longrightarrow$$

$$\frac{T}{\rho} = v^2 \rightarrow v = \sqrt{\frac{T}{\rho}}$$

Wave equation

$$F_y = T_y(x+dx) - T_y(x)$$

$$g = x \pm vt$$

$f(g)$ satisfy the equation

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial t} = \frac{df}{dg}\frac{\partial g}{\partial t} \quad / \, x$$

$$= \pm v \frac{df}{dg}$$

$$\frac{\partial f}{\partial x} = \frac{df}{dg}$$

$$\frac{\partial f}{\partial t} = \pm v \frac{\partial f}{\partial x}$$

$\pm v$: travelling wave

旅行的速度

“旅行的意义”

the solution of wave function:

$$y = A\cos(wt - kx) = A\cos w\left(t - \frac{x}{v}\right)$$

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2 \cos(wt - kx)$$

$$F_y = T_y(x + dx) - T_y(x)$$

$$= T\frac{\partial^2 y}{\partial x^2}dx$$

$$= -Tk^2 dx\, A\cos(wt - kx)$$

$$= -Tk^2 dx\, y$$

Hooke's law again

$$k_s = Tk^2 dx \quad (\text{coefficient of spring})$$

$$w^2 = \frac{k_s}{m} = \frac{Tk^2 dx}{\rho\, dx} = k^2 v^2$$

$$\left(v = \sqrt{\frac{T}{\rho}}\right)$$

k: wave number

$$k \equiv 2\pi/\lambda$$

$$w = 2\pi/T$$

波的传播速度: $v = \dfrac{\lambda}{T} = w/k$

$$(x, t) \longrightarrow (x+\Delta x,\ t+\Delta t)$$

$$\theta(x,t) = \theta(x+\Delta x,\ t+\Delta t)$$

$$\omega \Delta t = k \Delta x$$

$$\theta = \omega t - kx$$

$$d\theta = \frac{\partial \theta}{\partial t} dt - \frac{\partial \theta}{\partial x} dx$$

$$= \omega\, dt - k\, dx$$

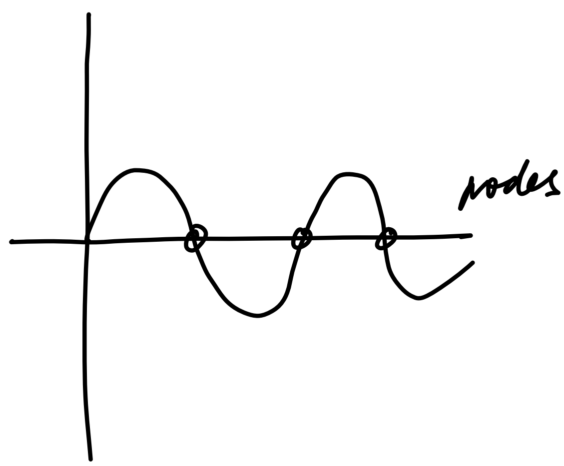
$$V_\theta = \left(\frac{dx}{dt}\right)_{[d\theta=0]} = \frac{\omega}{k}$$

相位传播速度

Crest 波峰

Trough /trɔːf/ 波谷

node 波节



关于 phase velocity

t_1 x_1 两个时空点

t_2 x_2

$\cos(\omega t - kx)$ $\theta = \omega t - kx$

$\theta_1 = \theta_2$, i.e.,

$$\omega t_1 - kx_1 = \omega t_2 - kx_2$$

$$d\theta = 0$$

$$dt = v\, dx$$

$V_p = V$ 相位变化

简谐波里，每个点的相位变化速度一样.

这一个点与整体讨论是一样的。

Another way :

$$[\sigma(x+\Delta x) - \sigma(x)]\, S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2}$$

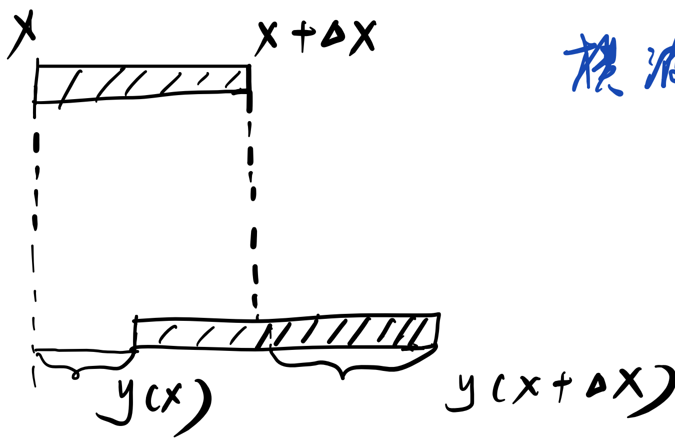
$$\frac{\partial \sigma}{\partial x}\cancel{\Delta x\, S} = \rho \cancel{S \Delta x} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}$$

在此讨论

横波 纵波

(疏密波)



$y(x)$:

X处质元偏离平衡位置位移

$\Delta y = y(x+\Delta x) - y(x)$: deformation

$\Delta y = \frac{\partial y}{\partial x}\Delta x$

(stress)

应力 杨氏模量

$$\frac{F}{S} = \sigma = -Y \frac{\Delta y}{\Delta x}$$

k_s deformation

$$F = -SY/\Delta x \; (\Delta y)$$

$$E_p = \frac{1}{2}\left(\frac{SY}{\Delta x}\right)\Delta y^2 = \frac{1}{2}\left(\frac{SY}{\Delta x}\right)\left(\frac{\partial y}{\partial x}\right)^2 \Delta x^2$$

$$= \frac{1}{2} Y S \Delta x \left(\frac{\partial y}{\partial x}\right)^2$$

$$\sigma(x+\Delta x) S - \sigma(x) S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} \Delta x \cancel{S} = \rho \cancel{S} \Delta x \frac{\partial^2 y}{\partial t^2}$$

here

$$Y \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2} \quad (\rho \text{ 是体密度})$$

$$V = \sqrt{\frac{Y}{\rho}}$$

$$y = A\cos(\omega t - kx) = A\cos\omega\left(t - \frac{x}{v}\right)$$

$$E_p = \frac{1}{2} Y S \Delta x \left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2} Y S \Delta x k^2 A^2 \sin^2(\omega t - kx)$$

$$E_k = \frac{1}{2} dm \left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2} \rho \Delta x S \omega^2 A^2 \sin^2(\omega t - kx)$$

$$E_p = E_k \Rightarrow E = E_k + E_p = \rho \Delta x S \omega^2 A^2 \sin^2(\omega t - kx)$$

(体积元)

能量密度 $\frac{E}{V} = \rho \omega^2 A^2 \sin^2(\omega t - kx)$

是时间的周期函数

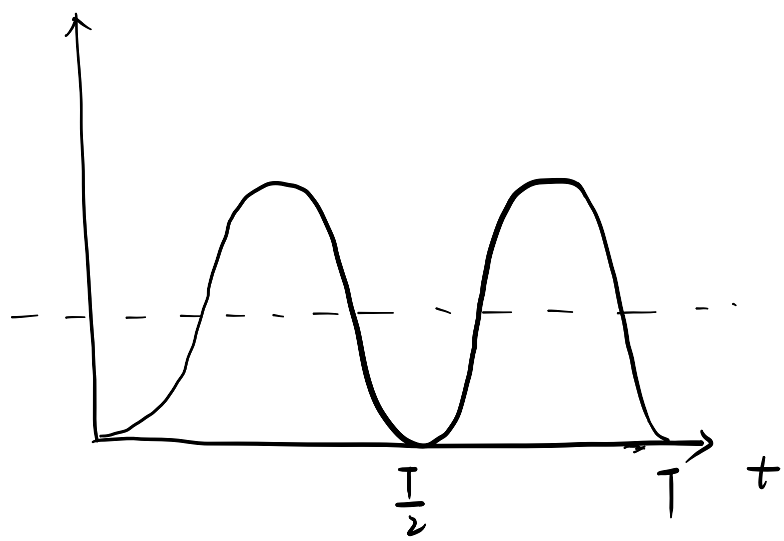
参照上面的推导：

每一个质元不是独立的弹簧，不会任何时刻独自能量守恒。

所有的弹簧之间彼此牵扯，各自在一个周期内经过人生的高低起落，平均能量相等。

$$E \propto (1-\cos 2(\omega t - kx))$$

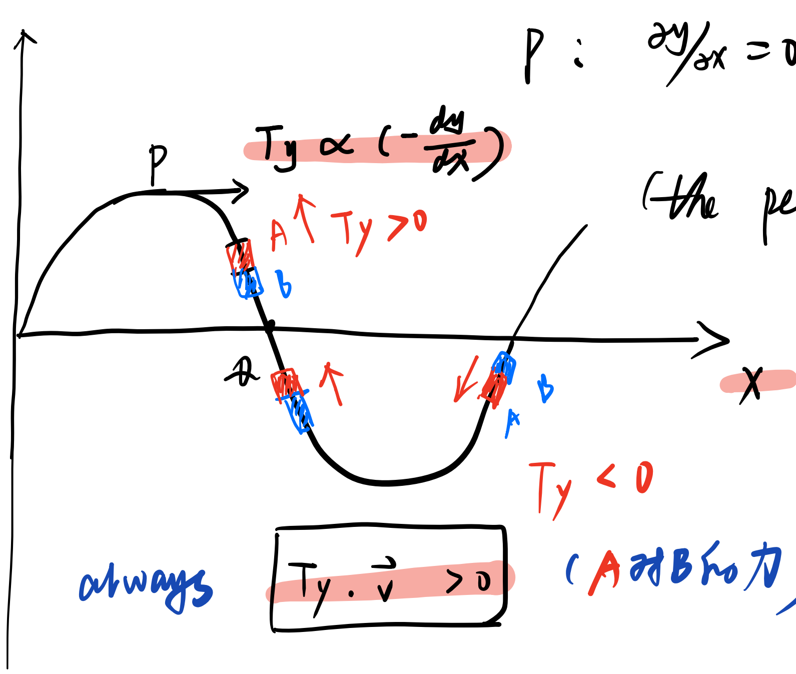
figure：



如果一个质元弹簧、时刻能量守恒，将不会有能量传播这样一件事情。右行波，左边质元一直对右边做正功。

$$y = A\cos(\omega t - kx)$$

Fig (a)



always $\boxed{T_y \cdot \vec{v} > 0}$ (A对B做功)

P: $\partial y/\partial x = 0$, deformation = 0

$E_p = 0$

(the peak of life, no velocity,

$E_k = 0$

$t_0 = \frac{T}{4} \Rightarrow y = A\cos(\frac{\pi}{2} - kx)$

$= A \sin kx$

(如左图)

P点 曲线 $\frac{\partial y}{\partial x}$ 同号,

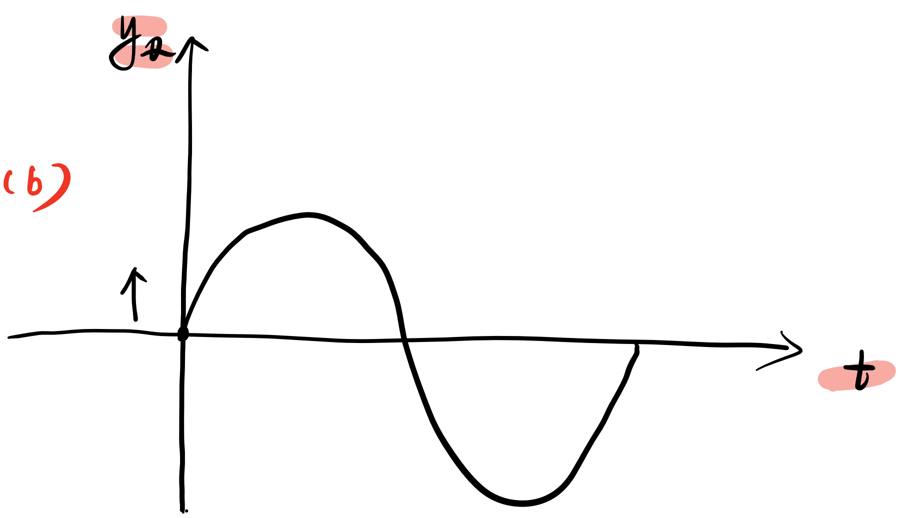
$\boxed{\begin{array}{l}\text{在你之前的质元} \\ \text{永远对你作正功}\end{array}}$

Q点: $t_0 = \frac{T}{4}$, $x = \frac{\lambda}{2}$

$$y_Q = A\cos(\frac{\pi}{2} - \pi + \omega t)$$

$$= A \sin \omega t$$

Fig (b)



$$\left[\begin{array}{c} \frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x} \\ \text{local dynamics} \\ \downarrow \\ \text{shape} \end{array} \right]$$

$\frac{\partial y}{\partial t}\Big|_{t=0}$ takes maximum

比较

上图 $\frac{\partial y}{\partial x}$ 也最大，deformation 最大

Fig(a)

同时最大

review the energy of the mass element :

$$W = \rho S \Delta x\, w^2 A^2 \sin^2(wt - kx)$$

$$\frac{W}{S\Delta x} = w = \rho w^2 A^2 \sin^2(wt - kx)$$

$\downarrow$

Energy density

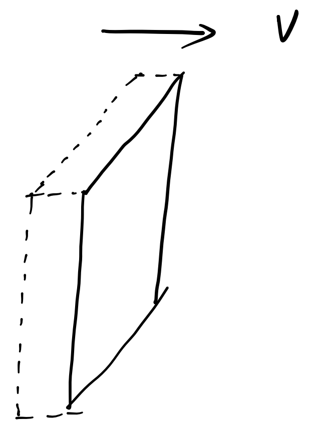
$$\langle w \rangle = \frac{1}{T}\int_0^T \rho w^2 A^2 \sin^2(wt - kx)\, dt$$

$$= \frac{1}{2} \rho w^2 A^2$$

travelling wave

signal transmit

$$v = \frac{\omega}{k}, \quad t = T$$



$v\,dt\,S\,w$ 能量密度

dt 时间，穿过截面 s 的能量

$$P = \frac{v\,dt\,s\,w}{dt}$$ 瓦（功率）

$$= \boxed{w\,S}\,v$$

线能量密度　　流起来.

比较电流强度

$I = nesv$

$$I = \frac{q}{t} = \frac{nsle}{t} = nesv$$

电荷流起来

$$\frac{\langle P\rangle}{S} = I = \langle w\rangle v \qquad [瓦/米^2]$$

体能流密度　　流起来

（波的强度）

一个周期内通过 S_1 和 S_2 的能量相等

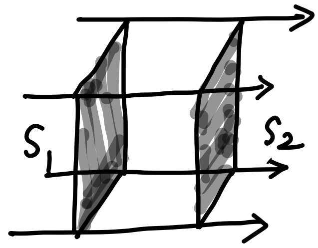
$$I_1 S_1 T = I_2 S_2 T$$

平面波 ： plane wave（等相面是平面）

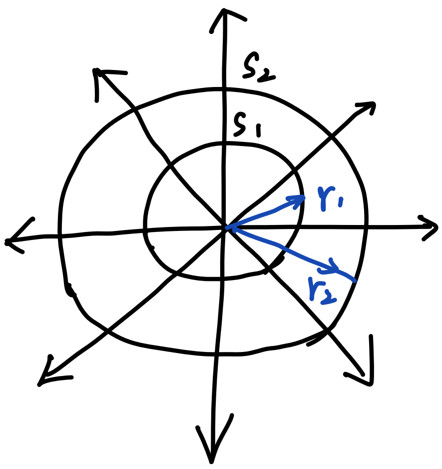
$$\frac{1}{2}\rho v w^2 A_1^2 s_1 T = \frac{1}{2}\rho v w^2 A_2^2 s_2 T$$

$S_1 = S_2$

$A_1 = A_2$



球面波 ： 等相面是球面



$$\frac{S_1}{r_1^2} = \frac{S_2}{r_2^2}$$

$$A_1^2 S_1 = A_2^2 S_2$$

$$\frac{S_1}{S_2} = \frac{r_1^2}{r_2^2} = \left(\frac{A_2}{A_1}\right)^2$$

$$A_2 r_2 = A_1 r_1$$

$$y = \frac{A_1}{r}\cos w\left(t - \frac{r}{v}\right)$$

一般情况下，会有波的吸收，转化成

介质的内能或热。

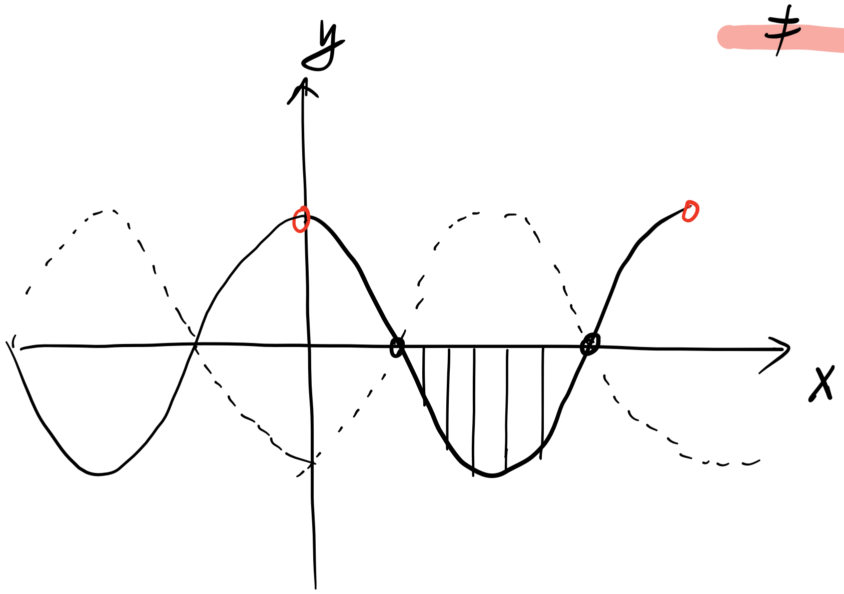
standing wave :

$$y = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

$$= 2A\cos\omega t \cos kx$$ no travelling

$$= 2A\cos kx \cos\omega t$$ $y(t+\Delta t, x+v\Delta t)$

$\neq y(t, x)$



amplitude : position dependent

birth rate

波节 : $E_k \equiv 0$

$\langle w \rangle v - \langle w \rangle v = 0$ $E_p + E_k = \text{const}$

without energy transfer

E(x) keep invariant at any time point

standing, what about the tension?

go back to the superposition of the left / right travelling waves

$$\langle W \rangle_L = \langle W \rangle_R$$

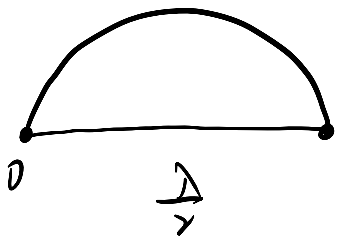
$$W_L(t) \neq W_R(t)$$

— reply to 常天硕's Q

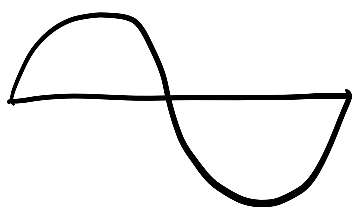
the superposition of two travelling waves

$\langle E_k \rangle = \langle E_p \rangle$

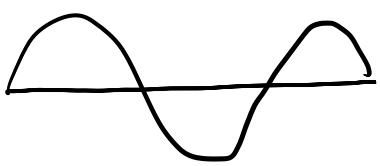
nodes:



$\frac{\lambda}{2} = L$



$\lambda = L$



$\frac{3}{2}\lambda = L$

$\sin kL = 0$

$kL = n\pi$

$k = n\frac{\pi}{L}$.

$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$

$f_n = \frac{\omega_n}{2\pi} = \frac{v k_n}{2\pi} \Leftarrow$

$= \frac{v}{2\pi} n \rightarrow$ integer

$y(x,t) = \sum_n C_n \sin k_n x \cos \omega_n t$

Fourier transformation $\rightarrow$ a special linear superposition

拍: $y = A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x)$

beat frequency

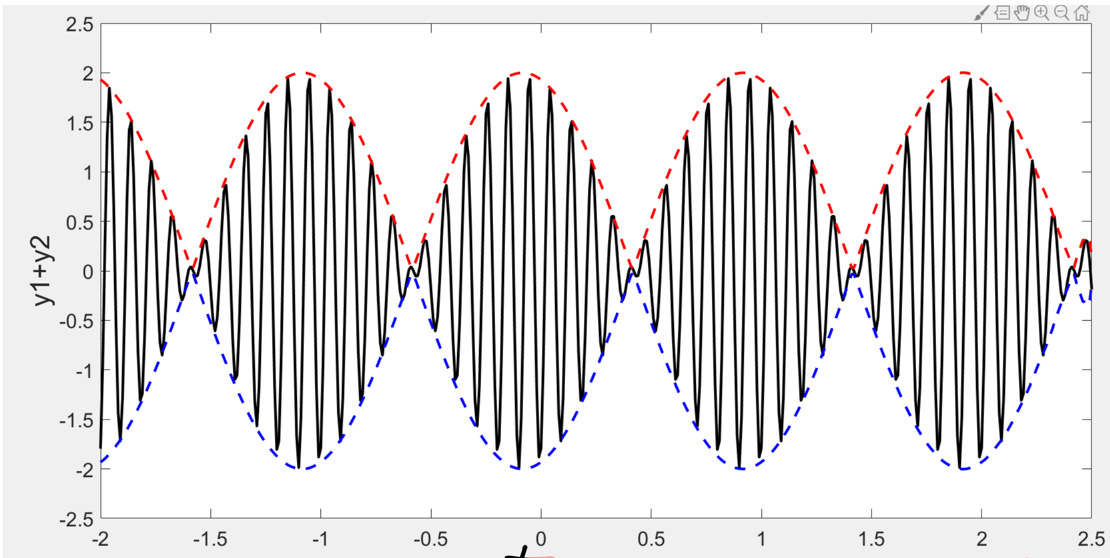
$$= 2A \cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right)$$

$$\cos\left(\frac{\omega_1-\omega_2}{2}t - \frac{k_1-k_2}{2}x\right)$$

$\omega_1 \approx \omega_2$

$$= A_g \cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right)$$

$x = 0.5$

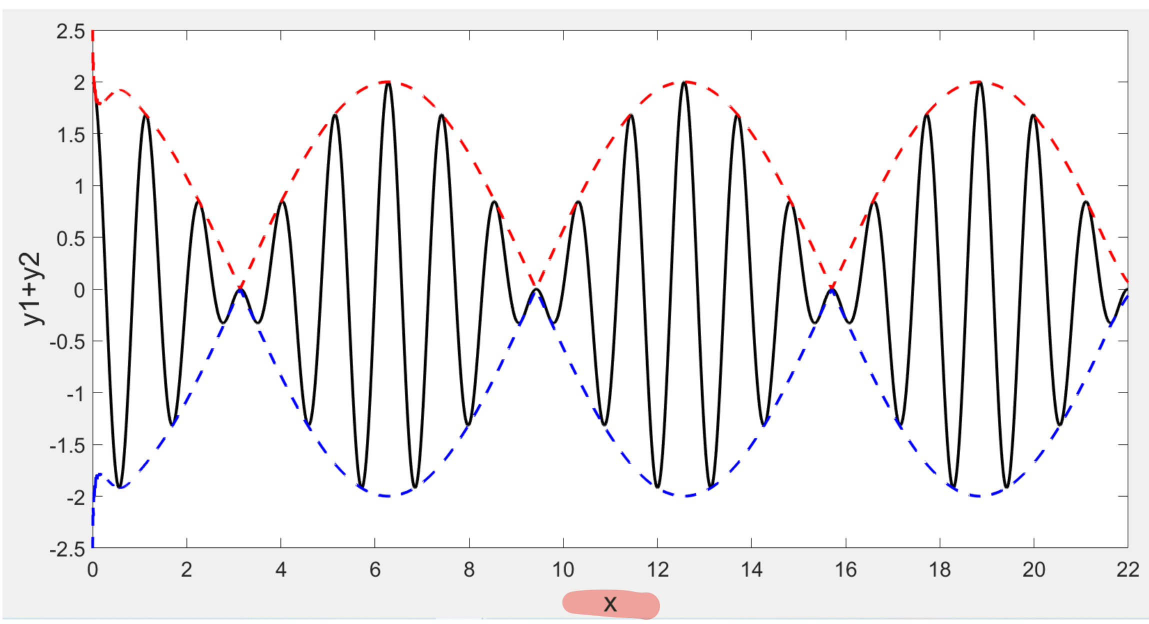


t

E.g. $\omega_1 = 10 \times 2\pi$ $\quad k_1 = 6$ $\quad A = 1$

$\omega_2 = 11 \times 2\pi$ $\quad k_2 = 5$

$t = 1$



travelling wave : "envelope" velocity

A_g 缓变 $w_g = \left|\frac{w_1 - w_2}{2}\right|$

质元以 $\bar{w} = \frac{w_1 + w_2}{2}$ 快变

这里: group : 波群, 波包, 包络线 (envelope)

$$V_g = \frac{w_1 - w_2}{k_1 - k_2} \approx \frac{dw}{dk}$$

$$V_p = \frac{w_1 + w_2}{k_1 + k_2} \approx \frac{w}{k}$$

Q: 听的是什么? 能听出 A_g, $-A_g$ 的差别吗?

听到的拍频 $f_{拍} = 2 \times \frac{|w_1 - w_2|}{2\pi}$

$$= |w_1 - w_2| / \pi$$