回顾转动用矢量描述的

"不可加性"，不可交换性

(有大小，有方向) 但不是矢量

已经是不平移的写法

$\vec{\theta}_1 \equiv \theta_1 \hat{\theta}_1$

$\vec{\theta}_2 \equiv \theta_2 \hat{\theta}_2$

Finite rotations are not vectors

(如果→用来约定矢量)

如果转轴相同 $\hat{\theta}_1 = \hat{\theta}_2$

$\vec{\theta}_1 + \vec{\theta}_2 = (\theta_1 + \theta_2)\hat{\theta}_1$ (平移后，是某种结合)

$= \vec{\theta}_2 + \vec{\theta}_1$

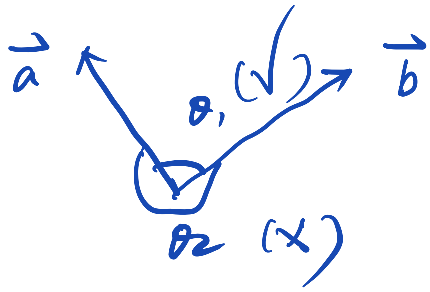
这里，θ，$\hat{\theta}_1$ 不能结合成矢量

对于一般情形： $\hat{\theta}_1 \neq \hat{\theta}_2$，转轴不同

(转动一个有限角度这个操作有转轴(方向)和转角(大小)两个要素，但二者并不能结合成一个矢量，自然不具有矢量的运算法则。以上是形式化的一个处理。)

($\vec{a}$, $\vec{b}$ define a plane)

使用右手 : rotate $\vec{a}$ into $\vec{b}$ through the $\boxed{\text{lesser}}$ of the two possible angles



$\vec{a} \times \vec{b}$

↓

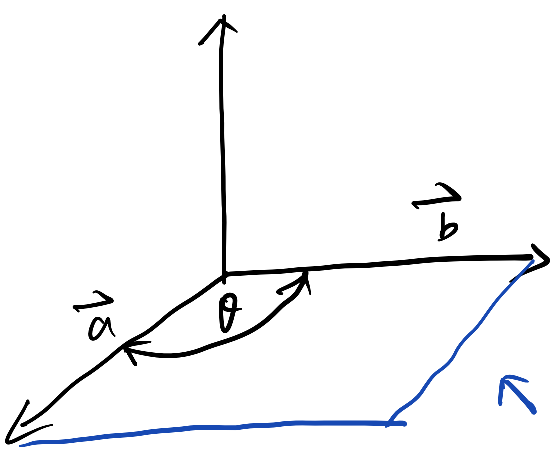
cross product (叉乘)

read as ($\vec{a}$ cross $\vec{b}$)

$$\vec{c} = \vec{a} \times \vec{b}, \quad \vec{c} \perp \vec{a}, \quad \vec{c} \perp \vec{b}$$

$$\vec{c} = \hat{c}\, ab\, |\sin(\vec{a}, \vec{b})|$$

$$|\vec{a} \times \vec{b}| = ab\, |\sin(\vec{a}, \vec{b})|$$



parallelogram

/ˌpærəˈleləgræm/

平行四边形

read as

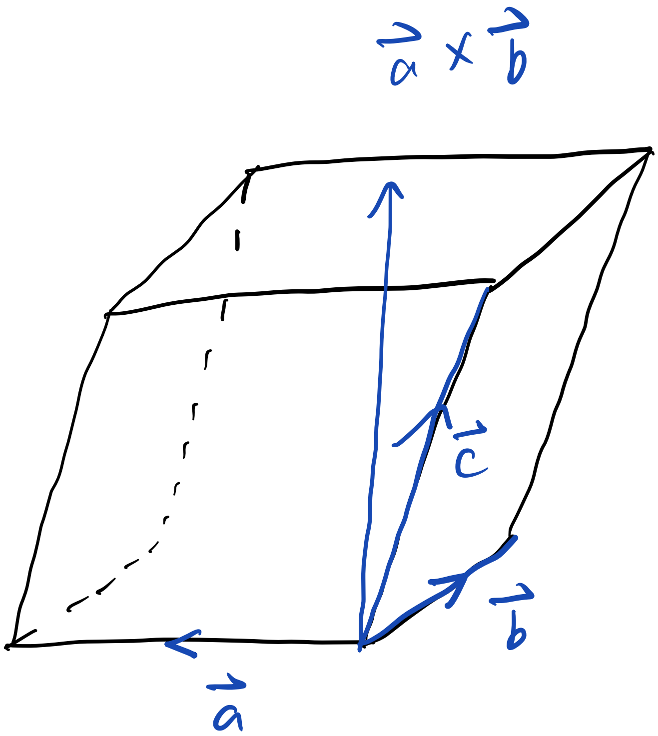
$\vec{a} \cdot \vec{b}$ ($\vec{a}$ dot $\vec{b}$)

↓ inner product (点乘)

$\vec{c} \cdot (\vec{a} \times \vec{b})$ ......

可能是你说的混合积

但这都是术语（也称"黑话"）



$$\left| (\vec{a} \times \vec{b}) \cdot \boxed{?} \right| = V$$

平行六面体

volume of parallelepiped

/ˌpærəˌlelə'paiped/

$\vec{c}$ 可以换向

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$$

$\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{x} = -\hat{z}$

$\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{y} = -\hat{x}$

Laws of sines

$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{a} \times \vec{c} = \vec{a} \times \vec{a} + \vec{a} \times \vec{b}$$

$$ac \sin(\vec{a}, \vec{c}) = ab \sin(\vec{a}, \vec{b})$$

$$\frac{\sin(\vec{a}, \vec{c})}{b} = \frac{\sin(\vec{a}, \vec{b})}{c}$$

回看维度

$\vec{a} \cdot \vec{b}$ $\qquad$ $\vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{b}$ $\qquad$ projection

$\vec{a} \times \vec{b} = ab\sin\theta_{\vec{a},\vec{b}}$

平行四边形面积

$\vec{a}$，$\vec{b}$ 共线

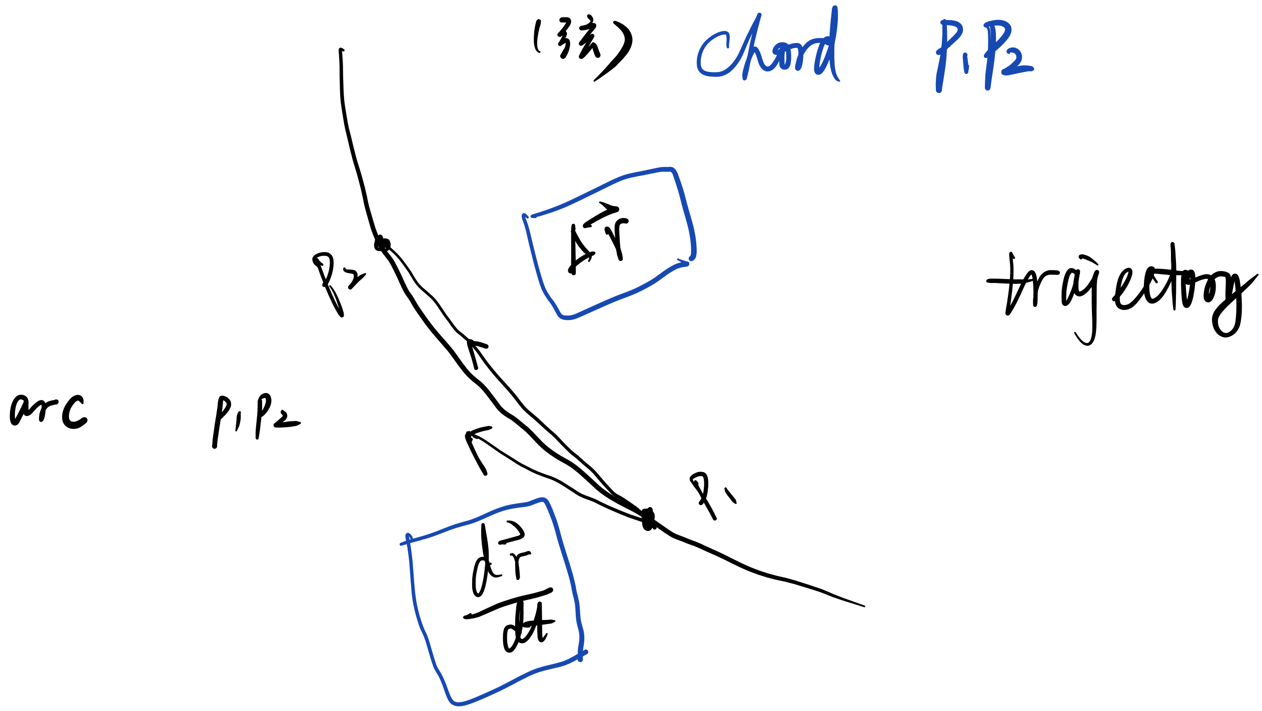
1° 没有平行四边形

只有一维，没有面

2° 更无法构成体

$(\vec{a} \times \vec{b}) \cdot \vec{c}$

没有平面，无从定义垂直于平面的直线。



理清概念

$$\frac{\Delta \vec{r}}{\Delta t} \xrightarrow{\Delta t \to 0} \frac{d\vec{r}}{dt}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt}$$

time derivative of $\vec{r}$

（此处强调平行四边形法则：矢量合成）

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$$

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

互相垂直的方向 "各走各路"

分量即投影，可以反过来强调内积。

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

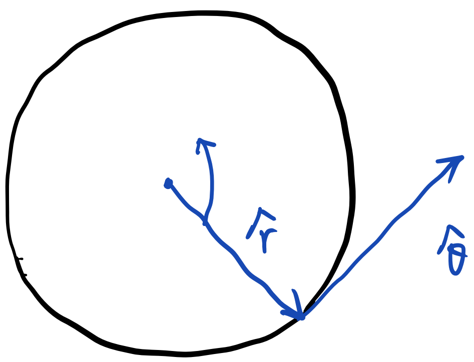
(unit vectors are fixed)

In general :

$$\vec{r}(t) = r(t)\ \hat{r}(t)$$

$$\frac{d\vec{r}}{dt} = \frac{dr(t)}{dt}\hat{r}(t) + r(t)\frac{d\hat{r}(t)}{dt}$$

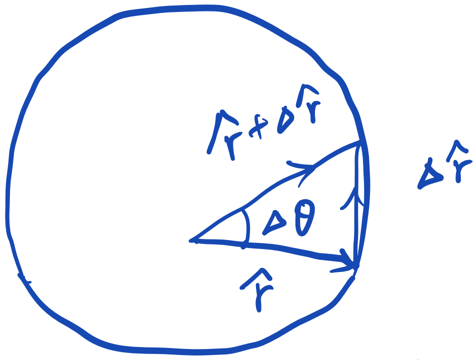
$\hat{r}$: the unit radial vector



$\hat{\theta}$: a unit vector perpendicular to $\hat{r}$

and in the direction of increasing θ

Circular path :



$$|\Delta\hat{r}| = |\hat{r}|\,\Delta\theta$$

$\Delta t \to 0$

$\Delta\theta \to 0$

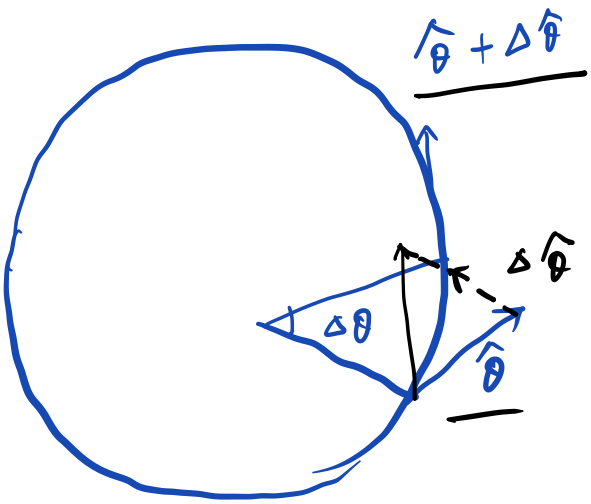
$\hat{r}$, $\hat{\theta}$: 方向单位矢量

$$\Delta\hat{r} = \Delta\theta\,\hat{\theta}$$

$$\frac{\Delta\hat{r}}{\Delta t} = \frac{\Delta\theta}{\Delta t}\hat{\theta}$$

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}$$

$$\boxed{\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}}$$



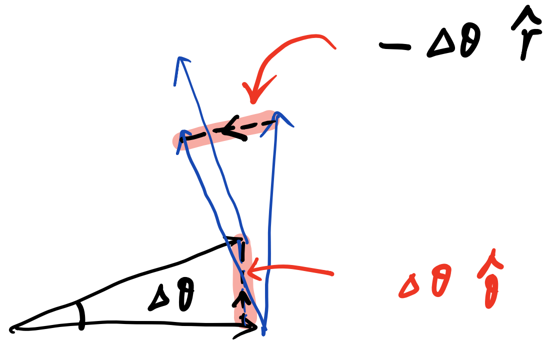
unit vector

unit vector

$\Delta\theta$ 太大，看不出一个这个方向

停一停：问产生方向

(此例子非常好地复习了微分的求和以及乘法规则)



generalize to the motion on any path

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}$$

$$= r\frac{d\theta}{dt}\hat{\theta} + \frac{dr}{dt}\hat{r}$$

勇敢者可以继续挑战

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(r\frac{d\theta}{dt}\hat{\theta}\right) + \frac{d}{dt}\left(\frac{dr}{dt}\hat{r}\right)$$

$$= \underbrace{\frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta}}_{①} + \underbrace{r\frac{d^2\theta}{dt^2}\hat{\theta}}_{②} + \underbrace{r\frac{d\theta}{dt}\left(\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r}\right)}_{③}$$

$$+ \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\left(\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}\right)$$

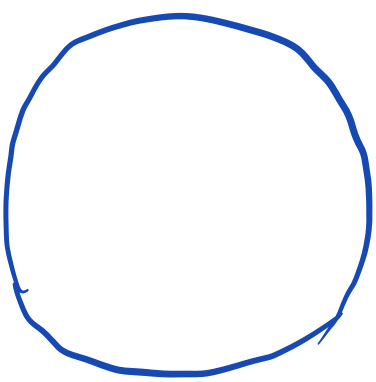
$$\quad 4 \qquad\qquad 5$$

合并同类项： 1 = 5

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r}$$

$$+ \frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\hat{\theta}$$

以匀速圆周运动为例



(r, θ)

$r(t) = r$

$\theta(t) = \omega t$

$\vec{v} = \omega r\hat{\theta}$

$\vec{a} = -\omega^2 r\hat{r}$

$(x, y) = (r\cos\theta, \quad r\sin\theta)$

利用 $(\cos\theta(t))' = -\sin\theta(t)\dfrac{d\theta(t)}{dt}$

$(\sin\theta(t))' = \cos\theta(t)\dfrac{d\theta}{dt}$

chain rule

自由度的对应：

$(r, \theta) \rightarrow (x, y)$ 　　2个

$\vec{v} = (\dot{x}, \dot{y}) = (-w\sin\theta, \ w\cos\theta)$

what is the direction ?

check the orthogonality

$\vec{r} = (x, y)$

$\vec{r} \cdot \vec{v} = 0$

$\vec{a} = (\ddot{x}, \ddot{y}) = (-w^2\cos\theta, -w^2\sin\theta)$

$= -w^2\vec{r}$ （向心加速度）

$\vec{a} \cdot \vec{v} = 0$

如果时间充裕，介绍 Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

补充平均速度

（不求极限即可）

$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t}$$